LONG TIME BEHAVIOUR OF A GENERAL CLASS OF BRANCHING MARKOV PROCESSES

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Abstract. We introduce a general class of branching Markov processes for the modelling of structured populations. Each individual has a nonnegative trait (size, quantity of parasites or proteins, ...), which evolves as a diffusion with positive jumps. The growth rate, diffusive function and jump rate of this trait may depend on the trait value. The individual death rate also depends on the trait. At death, an individual gives birth to two offspring and its trait is shared (unequally) between these two offspring. We study the long time behaviour of the trait along a lineage and in the whole population.

Key words and phrases: Continuous-time and space branching Markov processes, structured population, long time behaviour

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Introduction

We introduce a general class of continuous-time and space branching Markov processes for the study of structured populations. Each individual in the population is characterized by a trait whose dynamics follows a Markov process. Then, at a random time depending on the individual's trait dynamics, the individual dies and gives birth to two descendants, whose traits at birth depend on the trait of the mother. We are interested in the long time behaviour of these structured branching processes and in their biological interpretation in terms of population dynamics. First, we focus on the dynamics of the process along a lineage. We give criteria for the process along a lineage to be almost surely absorbed at 0 in finite time, to converge to a positive random variable, or to follow some long time asymptotics. Moreover, in the case of almost sure extinction, we give bounds on the exponential decay of the survival probability and in the case of survival with a positive probability, we prove that the process grows at least exponentially. Second, we introduce a one-dimensional auxiliary process, also known as spinal process, whose dynamics provides useful information on the dynamics of the whole structured population, via a Many-to-One formula (see [2] for instance). Combining couplings and results derived in the study of the process along a lineage, we investigate the long time behaviour of this auxiliary process and prove that it converges under some assumptions on the trait dynamics. Moreover, we show that this auxiliary process corresponds to the trait of a uniformly sampled individual in the population and deduce some properties on the long time behaviour of the process at the population level, extending previous results derived for a smaller class of structured Markov branching processes (see [6, 2, 10] for instance).

Notice that the study of the process along a lineage is of independent interest. It belongs to a class of processes recently introduced as strong solutions of Stochastic Differential Equations (SDE) in [30], and only few processes of this class have been studied until now. This class of processes allows to take into account interactions between individuals as well as the effects of the environment. The addition of interactions between individuals in continuous-time branching processes has recently attracted a lot of interest. For instance, Feller diffusions and Continous State Branching Processes (CSBP) with logistic competition have been studied in [22, 24] and [7], respectively, Feller diffusions with some nonlinear birth rates have been studied in [33], and polynomial interactions have been considered in [25]. Li and coauthors [26] have recently introduced a general class of continuous state nonlinear branching processes, and have investigated extinction, explosion and coming down from infinity for this class. However, in all these models, only positive jumps are allowed. They result from large birth events, and were first introduced by constructing the continuous state process as the limit of a sequence of discrete branching processes (see for instance [23, 5]). In parallel, models where the interactions between individuals result from the fact that the whole population is subject to the variations of the same environment have been intensively studied recently, in particular in the framework of CSBPs in random environment. This class of models, initially introduced by Keiding and Kurtz [21, 20] in the case of Feller diffusions in a Brownian environment, have been recently generalised and studied by many authors [9, 3, 32, 31, 17, 30, 27, 4]. In this setting, negative jumps may occur, being for instance the result of environmental catastrophes killing each individual with the same probability [3]. However, in CSPBs in random environment, the environment is independent of the population state. In particular the rate of catastrophes does not depend on the population size. We relax this assumption in the current work.

A classical method to obtain information on the distribution of one trait in a structured branching population is to introduce a spinal decomposition and to prove a Many-to-One formula. It consists in distinguishing a particular line of descent in the population, and to prove that the dynamics of the trait along this particular lineage is representative of the dynamics of the trait of a typical individual in the population, *i.e.* an individual picked uniformly at random. In particular, we refer to [14, 16, 10] for general results on these topics in the continuous-time case. Using recent results of Marguet in [28], we exhibit an auxiliary process for the class of processes under consideration, in the particular case where the division rate is constant or linear in the individual's trait. Then, to obtain properties at the population level, we study this (time-inhomogeneous) auxiliary process. In particular, we prove its convergence, using results in [29], and extending some results of the first part of our study (concerning the process along a lineage) to the time-inhomogeneous case.

The main application that we consider for our study is the modelling of a parasite infection in a cell population. Some experiments, conducted in the TAMARA laboratory, have shown that cells distribute unequally their parasites between their two daughter cells [36]. This could be a mechanism aiming at concentrating the parasites in some cell lines in order to "save" the remaining lines. It is thus important to understand the effect of this unequal sharing on the long time behaviour of the infection in the cell population. This question has been addressed by Bansaye and Tran in [6]. They introduced and studied branching Feller diffusions with a cell division rate depending on the quantity of parasites in the cell and a sharing of parasites at division between the two daughter cells according to a random variable with any symmetric distribution on $[0, 1]$. They provided some extinction criteria for the infection in a cell line, in the case where the cell division rate is a constant or monotone function of the quantity of parasites in a cell, as well as recovery criteria at the population level, in the constant division rate case. In [3], Bansaye and coauthors extended this study by providing the long time asymptotic of the recovery rate in the latter case. Our work further extends these results in several directions. First, we allow the parasites growth rate and diffusion coefficient in a cell to depend on the quantity of parasites. Second, we add the possibility to have positive jumps in the parasites dynamics, with a rate which may depend on the quantity of parasites. Finally, we do not make any assumption on the monotonicity of the cell division rate. This situation is much more difficult to study than the previous ones, as the genealogical tree of the cell population depends on the whole history of the number of parasites in the different cell lines, whereas in [6, 3], the growth of the population is independent of the parasites dynamics. Another application we can think of, similar in spirit, is the modelling of the protein aggregates in a cell population. These latter, usually eliminated by the cells, can undergo sudden increases due to cellular stress for instance (positive jumps), and are known to be distributed unequally between daughter cells (see [35] for instance).

The paper is structured as follows. The class of branching Markov processes under consideration is introduced in Section 1. In Section 2, we give results on the long time behaviour of the process along a lineage. Section 3 is dedicated to the results on the long time behaviour of the process at the population level. The proofs are derived in Section 4.

In the sequel $\mathbb{N} := \{0, 1, 2, ...\}$ will denote the set of nonnegative integers, $\mathbb{R}_+ := [0, \infty)$ the real line and $\mathbb{R}_+^* := (0, \infty)$. We will denote by $\mathcal{C}_b^2(\mathbb{R}_+)$ the set of twice continuously differentiable bounded functions on $\mathbb{R}_+$. Finally, for any stochastic process X on $\mathbb{R}_+$ or Z on the set of point measures on $\mathbb{R}_+$, we will denote by $\mathbb{E}_x\left[f(X_t)\right] = \mathbb{E}\left[f(X_t)\middle|X_0 = x\right]$ and $\mathbb{E}_{\delta_x}\left[f(Z_t)\right] = \mathbb{E}\left[f(Z_t)\middle|Z_0 = \delta_x\right]$.

1. Definition of the population process

We use the classical Ulam-Harris-Neveu notation to identify each individual. Let

$$\mathcal{U} := \bigcup_{n\in\mathbb{N}} \{0,1\}^n,$$

denote the set of all possible labels.

In this section, we define the structured population model, following the construction detailed in [28]. We consider a branching Markov process where each individual in the population is characterized by a trait. We assume that this trait evolves as a diffusion with positive jumps. More precisely, we consider the SDE

$$\mathfrak{X}_t = \mathfrak{X}_0 + \int_0^t g(\mathfrak{X}_s)\mathfrak{X}_s ds + \int_0^t \sqrt{2\sigma^2(\mathfrak{X}_s)}dB_s + \int_0^t \int_0^{p(\mathfrak{X}_{s-})} \int_{\mathbb{R}_+} z\widetilde{Q}(ds, dx, dz), \tag{1.1}$$

where $\mathfrak{X}_0$ is nonnegative, g, σ and p are nonnegative functions on $\mathbb{R}_+$, B is a standard Brownian motion, $\widetilde{Q}$ is a compensated Poisson point measure with intensity $ds \otimes dx \otimes \pi(dz)$ independent of B and π is a probability measure on $\mathbb{R}_+$.

An individual with trait x dies at rate $r(x)$ where r is a nonnegative function on $\mathbb{R}_+$ and is replaced by two individuals with characteristics at birth given by Θx and $(1-\Theta)x$ where Θ is a nonnegative random variable on $[0,1]$ such that $\mathbb{P}(\Theta = 0) = \mathbb{P}(\Theta = 1) = 0$ and with associated symmetric distribution $\kappa(d\theta)$.

We denote by $\mathcal{M}_P(\mathbb{R}_+)$ the set of point measures on $\mathbb{R}_+$. Following [12], we work in $\mathbb{D}(\mathbb{R}_+, \mathcal{M}_P(\mathbb{R}_+))$, the set of càdlàg measure-valued processes. For any $Z \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_P(\mathbb{R}_+))$, $t \geq 0$, we write

$$Z_t = \sum_{u \in V_t} \delta_{X_t^u}, \tag{1.2}$$

where $V_t \subset \mathcal{U}$ denotes the set of all individuals alive at time t and for all $u \in V_t$, X_t^u denotes the trait at time t of the individual u. Some assumptions are needed to ensure that the process is well-defined. The first set of hypotheses ensures the existence and uniqueness of a strong solution to (1.1) (see the proof of Proposition 2.1 for details).

Assumption A. - *The functions r, p and g are continuous, and for any $n \in \mathbb{N}$ there exists a finite constant B_n such that for any $0 \leq x \leq y \leq n$*

$$|yg(y) - xg(x)| \leq B_n(y-x)\left(\ln(y-x)^{-1} + 1\right).$$

- *The function p is locally Lipschitz and $p(0) = 0$.*
- *The function σ is 1/2-Hölderian on compact sets and $\sigma(0) = 0$.*
- *The measure π satisfies*

$$\int_0^\infty \left(z \wedge z^2\right) \pi(dz) < \infty.$$

Then according to [30, Proposition 1], there is a unique strong solution $(\mathfrak{X}_t, t \geq 0)$ to (1.1). It is a Markov process with infinitesimal generator $\mathcal{G}$, satisfying for all $f \in C_b^2(\mathbb{R}_+)$,

$$\mathcal{G}f(x) = xg(x)f'(x) + \sigma^2(x)f''(x) + p(x) \int_{\mathbb{R}_+} \left(f(x+z) - f(x) - zf'(x)\right) \pi(dz).$$

Following [28], we denote by $(\Phi(x,s,t), s \leq t)$ the corresponding stochastic flow *i.e.* the unique strong solution to (1.1) satisfying $\mathfrak{X}_s = x$ and the dynamics of the trait between division events is well-defined.

We also need assumptions to ensure the non-explosion of the process in finite time.

Assumption B. *i) There exist $r_1, r_2 \geq 0$ and $\gamma \geq 1$ such that for all $x \geq 0$*

$$r(x) \leq r_1 x^\gamma + r_2.$$

ii) There exist $c_1, c_2 \geq 0$ such that, for all $x \in \mathbb{R}_+$,

$$\lim_{n \to +\infty} \mathcal{G}h_{n,\gamma}(x) \leq c_1 x^\gamma + c_2,$$

where γ has been defined in i) and $h_{n,\gamma} \in C_b^2(\mathbb{R}_+)$ is a sequence of functions such that $\lim_{n\to+\infty} h_{n,\gamma}(x) = x^\gamma$ for all $x \in \mathbb{R}_+$.

We can now properly define the population process Z. Let $E = \mathcal{U} \times [0,1] \times \mathbb{R}_+$ and $N(ds, du, dz, d\theta)$ be a Poisson point measure on $\mathbb{R}_+ \times E$ with intensity $ds \times n(du) \times dz \times \kappa(d\theta)$, where $n(du)$ denotes the counting measure on $\mathcal{U}$. Let $(\Phi^u(x,s,t), u \in \mathcal{U}, x \in \mathcal{X}, s \leq t)$ be a family of independent stochastic flows satisfying (1.1) describing the individual-based dynamics. We assume that M and $(\Phi^u, u \in \mathcal{U})$ are independent. We denote by $\mathcal{F}_t$ the filtration generated by the Poisson point measure M and the family of stochastic processes $(\Phi^u(x,s,t), u \in \mathcal{U}, x \in \mathcal{X}, s \leq t)$ up to time t.

Proposition 1.1. *Under Assumptions A and B, there exists a strongly unique $\mathcal{F}_t$-adapted càdlàg process $(Z_t, t \geq 0)$ taking values in $\mathcal{M}_P(\mathbb{R}_+)$ such that for all $f \in C_b^2(\mathbb{R}_+)$ and $t \geq 0$,*

$$\begin{aligned}\langle Z_t, f\rangle &= f(x_0) + \int_0^t \int_{\mathbb{R}_+} \mathcal{G}f(x) Z_s(dx)\, ds + M_t^f(x) \\ &+ \int_0^t \int_E \mathbf{1}_{\left\{u \in V_{s-},\ z \leq r(X^u_{s^-})\right\}} \left(f(\theta X^u_{s^-}) + f((1-\theta)X^u_{s^-}) - f(X^u_{s^-})\right) M(ds, du, d\theta, dz),\end{aligned}$$

where for all $t \geq 0$, $M_t^f(x)$ is a $\mathcal{F}_t$-martingale.

The proof is a direct application of [28, Theorem 2.2].

2. Behaviour of the process along a lineage

To understand the long time behaviour of the population, we will first focus on the process $(X_t, t \geq 0)$ which describes the dynamics of the process Z in a lineage, for example the most left lineage, and can be described by the SDE:

$$\begin{aligned}X_t = X_0 + \int_0^t g(X_s)X_s ds + \int_0^t \sqrt{2\sigma^2(X_s)} dB_s + \int_0^t \int_0^{p(X_{s^-})} \int_{\mathbb{R}_+} z\widetilde{Q}(ds, dx, dz) \qquad (2.1)\\ + \int_0^t \int_0^{r(X_{s^-})} \int_0^1 (\theta - 1) X_{s^-} N(ds, dz, d\theta),\end{aligned}$$

where $\widetilde{Q}$ and B have been introduced in (1.1), and N is a Poisson random measure with intensity $ds \otimes dz \otimes \kappa(d\theta)$, issued from M by choosing (arbitrarily) that any division events affects the particle on the left of the genealogical tree. Recall that by definition, N, $\tilde{Q}$ and B are mutually independent.

2.1. **Solution of the stochastic differential equation.** First of all, we state that under some mild conditions, the SDE (2.1) has a unique pathwise strong solution. We will work under these conditions in the sequel.

Assumption C. *Assumption A holds and for any* $0 \leq x \leq y \leq n$

$$\mathbf{1}_{\{r(x)<r(y)\}}(yr(y) - xr(x)) + \mathbf{1}_{\{r(y)\leq r(x)\}}(yr(y) + xr(x) - 2xr(y)) \\ \leq B_n(y-x)\left(\ln(y-x)^{-1} + 1\right).$$

The form of this assumption comes from the conditions of Proposition 1 in [30] that we will use to get the next result.

Proposition 2.1. *Suppose that Assumption C holds. Then, Equation* (2.1) *has a pathwise unique nonnegative strong solution.*

In the proofs of Section 3, we also need to consider a slight generalization of the SDE (2.1) where an individual with trait x dies and transmits a proportion $\theta \in [0,1]$ of its trait to its left offspring at a rate $r(x)l(\theta)$, that depends on θ, where $l : [0,1] \to \mathbb{R}_+$ is a nonnegative function. However, using the properties of Poisson random measures we can prove that a solution to such an SDE can be rewritten as the solution to (2.1) by modifying the death rate r and the fragmentation kernel κ.

Lemma 2.2. *Assume that* $\int_0^1 l(\theta)\kappa(d\theta) < \infty$. *Let*

$$\hat{\kappa}(d\theta) = l(\theta)\left(\int_0^1 l(\theta)\kappa(d\theta)\right)^{-1}\kappa(d\theta), \quad \hat{r}(x) = r(x)\int_0^1 l(\theta)\kappa(d\theta),$$

and $\widetilde{Q}$, B *and* N *be defined as in* (2.1). *Then, there exists a Poisson random measure* N' *with intensity* $ds \otimes dz \otimes \hat{\kappa}(d\theta)$ *such that* X *is the pathwise unique solution to*

$$X_t = X_0 + \int_0^t g(X_s)X_s ds + \int_0^t \sqrt{2\sigma^2(X_s)}dB_s + \int_0^t \int_0^{p(X_{s^-})} \int_{\mathbb{R}_+} z\widetilde{Q}(ds,dx,dz) \\ + \int_0^t \int_0^{\hat{r}(X_{s^-})} \int_0^1 (\theta-1)X_{s^-}N'(ds,dz,d\theta),$$

if and only if X *is the pathwise unique nonnegative strong solution to*

$$X_t = X_0 + \int_0^t g(X_s)X_s ds + \int_0^t \sqrt{2\sigma^2(X_s)}dB_s + \int_0^t \int_0^{p(X_{s^-})} \int_{\mathbb{R}_+} z\widetilde{Q}(ds,dx,dz) \\ + \int_0^t \int_0^{r(X_{s^-})\int_0^1 l(\theta)} (\theta-1)X_{s^-}N(ds,dz,d\theta).$$

The next two subsections are dedicated to the study of the long time behaviour of the process X solution to (2.1).

2.2. **Absorption probability and convergence of the process.** The first question we are interested in is to know if the trait reaches 0 along a lineage. It depends on the behaviour of the function σ, controlling the diffusive part of the process, in the neighbourhood of 0.

Let us introduce the stopping times $\tau^-(x)$ and $\tau^+(x)$ via

$$\tau^-(x) := \inf\{t \geq 0 : X_t < x\}, \quad \tau^+(x) := \inf\{t \geq 0 : X_t > x\}, \quad \text{for } x > 0 \tag{2.2}$$

and

$$\tau^{-}(0) := \inf\{t \geq 0 : X_t = 0\}, \tag{2.3}$$

with the convention $\inf \emptyset := \infty$.

To study the absorption of the process X, we will extend [26, Theorem 2.3]. This extension relies on the construction of a sequence of martingales satisfying the following assumption.

Assumption D. *Let $0 < c < \varepsilon < b$, $T = \tau^{-}(c) \wedge \tau^{+}(b)$. For some $a \in \mathbb{R}_+ \setminus \{1\}$ there exists a function G_a such that*

$$M_t^{(a)} := X_{t\wedge T}^{1-a} \exp\left(\int_0^{t\wedge T} G_a(X_s)ds\right)$$

is a martingale and

$$\mathbb{E}_\varepsilon\left[X_T^{1-a}\exp\left(\int_0^T G_a\left(X_s\right)ds\right)\right] \leq \varepsilon^{1-a}.$$

Moreover, in order to be able to use coupling arguments, we consider an additional assumption on the rate p.

Assumption E0. *The function p is nondecreasing.*

We can now state the result on the absorption of the process.

Theorem 2.3. *Suppose that Assumption E0 is satisfied. Let X be the pathwise unique solution to* (2.1)*, $0 < c < \varepsilon < b$ and $T = \tau^{-}(c) \wedge \tau^{+}(b)$.*

i) If there exist a nonnegative nondecreasing function f going to ∞ at ∞, and $a > 1$ such that Assumption D holds and

$$G_a(u) \geq -\ln(u^{-1})/f^2(1/u) \tag{2.4}$$

for u small enough, then $\mathbb{P}_x\left(\tau^{-}(0) < \infty\right) = 0$ for all $x > 0$.

ii) If there exist $a < 1$ and $\eta > 0$ such that Assumption D holds and

$$G_a(u) \geq \ln(u^{-1})\left(\ln(\ln(u^{-1}))\right)^{1+\eta} \tag{2.5}$$

for u small enough, then $\mathbb{P}_x\left(\tau^{-}(0) < \infty\right) > 0$ for all small enough $x > 0$.

As Theorem 2.3 extends [26, Theorem 2.3], the proofs of these two results are similar. However, several adaptations are needed as negative jumps may occur in our process. Moreover, we give tighter bounds than in [26, Theorem 2.3] where $(\ln(u^{-1}))^{r/2}, r < 1$ and $(\ln(u^{-1}))^{r/2}, r > 1$ were considered instead of the right-hand sides of (2.4) and (2.5).

In order to translate conditions 2.4 and 2.5 in terms of the functions controlling the dynamics of X, we now construct functions G_a satisfying Assumption D. This requires some conditions on the measure π and on the function p. Therefore, we slightly modify Assumption E0 to take into account those requirements.

Assumption E. *The function p is nondecreasing,* $\limsup_{0+} p(x)/x < \infty$ *and*

$$\int_0^\infty z\pi(dz) < \infty.$$

For the sake of readability, we consider Assumption E instead of a weaker assumption that would have been sufficient (we just need $G_a(x)$ to be well defined for small x) but harder to state. Next, we define

$$\mathcal{A} = \left\{a \in \mathbb{R}_+,\ \int_0^1 \left(\theta^{1-a} - 1\right)\kappa(d\theta) < \infty.\right\}$$

The set of functions G_a, for $a \in \mathcal{A}$, may then be introduced for $x > 0$ via

$$G_a(x) := (a-1)g(x) - a(a-1)\frac{\sigma^2(x)}{x^2} - r(x)\int_0^1 \left(\theta^{1-a} - 1\right)\kappa(d\theta)$$
$$- p(x)\int_{\mathbb{R}_+} \left((zx^{-1}+1)^{1-a} - 1 - (1-a)zx^{-1}\right)\pi(dz). \quad (2.6)$$

As an application of Theorem 2.3, we are now able to exhibit sufficient conditions on the diffusive term σ of the dynamics of X ensuring absorption.

(A1) There exist $a > 1 \in \mathcal{A}$, a nonnegative nondecreasing function f going to ∞ at ∞, and $u_0 > 0$ such that for all $u \leq u_0$

$$\frac{\sigma(u)}{u} \leq \left(\ln(u^{-1})\right)^{1/2} / f(1/u). \quad (2.7)$$

(A2) There exist $\eta > 0$ and $u_0 > 0$ such that for all $u \leq u_0$

$$\frac{\sigma(u)}{u} \geq \left(\ln(u^{-1})\right)^{1/2}\left(\ln(\ln(u^{-1}))\right)^{(1+\eta)/2}. \quad (2.8)$$

We can now state results on the absorption of the process in terms of those two conditions.

Corollary 2.4. *Suppose that Assumptions C and E hold.*

i) If Condition (A1) holds, then $\mathbb{P}_x(\tau^-(0) < \infty) = 0$ *for all* $x > 0$.
ii) If Condition (A2) holds then $\mathbb{P}_x(\tau^-(0) < \infty) > 0$ *for* x *small enough.*
iii) If Condition (A2) holds and moreover, $\sigma(x) + r(x) > 0$ *for any* $x > 0$*, then for any* $x > 0$ *and* $s > 0$, $\mathbb{P}_x(\tau^-(0) < s) > 0$.

The third point of Corollary 2.4 refines the result on the probability of non-absorption by stating that the probability to reach 0 at any positive time is positive under mild conditions. This property will be useful in the study of the process at the population level.

The long time behaviour of the process X in a lineage depends on the interplay between g, which tends to increase it, r, which decreases it, and the fragmentation kernel κ which has a less intuitive effect. From now on, Θ will denote a random variable distributed according to κ. We will always assume that the following condition holds:

$$|\mathbb{E}[\ln\Theta]| = \left|\int_0^1 \ln\theta\kappa(d\theta)\right| < \infty. \quad (2.9)$$

We consider the following possibilities for the relative strengths of g and r:

(B0) Condition (2.9) holds and there exist $\eta > 0$ and $x_0 \geq 0$ such that

$$g(x) + r(x)\mathbb{E}\left[\ln\Theta\right] \leq -\eta, \quad \forall\, x \geq x_0.$$

(B1) Condition (2.9) holds and there exist $\underline{r} > 0$ and $\eta \geq 0$ such that $r(x) \geq \underline{r}$, $\forall x \geq 0$ and

$$\frac{g(x)}{r(x)} + \mathbb{E}\left[\ln\Theta\right] \leq -\eta, \quad \forall\, x \geq 0.$$

(B2) Condition (2.9) holds and there exist $\underline{r} > 0$ and $\eta > 0$ such that $r(x) \geq \underline{r}$, $\forall x \geq 0$ and

$$\frac{g(x)}{r(x)} + \mathbb{E}\left[\ln\Theta\right] \geq \eta, \quad \forall\, x \geq 0.$$

The next result states that under Condition (B0), the division mechanism overcomes the growth of X. In this case, the process X converges to a finite variable, which may be 0 if X can be absorbed. To state this result, we need to introduce the following sequence of stopping times. Let $t_0 > 0$, $T_0 = 0$ and for all $i \geq 1$ and $x_0 \geq 0$,

$$T_i(x_0) = \inf\{t \geq T_{i-1}(x_0) + t_0,\ X_t \leq x_0\}. \quad (2.10)$$

Theorem 2.5. *Suppose that Assumptions C, E and Condition (B0) hold for some $x_0 > 0$. Then, for all $x \geq 0$, the process $(X_t, t \geq 0)$ converges in law as t tends to infinity to X_∞ with distribution*

$$\mathbb{P}(X_\infty \in \cdot) = \frac{1}{\mathbb{E}[T_1(x_0)]}\mathbb{E}\left[\int_0^{T_1(x_0)} \mathbf{1}_{\{X_s \in \cdot\}} ds\right], \tag{2.11}$$

satisfying

$$\mathbb{E}\left[X_\infty\left(g(X_\infty) - \frac{r(X_\infty)}{2}\right)\right] = 0. \tag{2.12}$$

Moreover, the distribution of X_∞ is the unique stationary distribution of the process X and for every bounded and measurable function f, almost surely,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X_s)ds = \mathbb{E}[f(X_\infty)].$$

Furthermore, if Condition (A2) holds and there exists $\mathfrak{d} > 0$ such that $\inf_{0\leq y\leq x_0+\mathfrak{d}} r(y) + \sigma(x) > 0$ for all $x > 0$, then $X_\infty = 0$ almost surely and for all $x \geq 0$,

$$\mathbb{P}_x\left(\exists t < \infty, X_t = 0\right) = 1. \tag{2.13}$$

2.3. **Classification of the long time behaviours.** Finally, we provide some properties on the long time behaviour of X under Assumptions (B1) and (B2) with $x_0 = 0$, extending the classification for stable CSPBs with random catastrophes (corresponding to $\sigma(x) = \sigma\sqrt{x}$ and $\pi \equiv 0$ or $\sigma \equiv 0$, $p(x) = x$ and π stable, $r(x) = r$ and $g(x) = g$) given in [3, Corollary 2].

Proposition 2.6. *Suppose that Assumptions C and E are satisfied.*

i) If Condition (B1) holds for $\eta > 0$, then

$$\lim_{t\to\infty} X_t = 0, \quad \textit{almost surely.}$$

ii) If Condition (B1) holds for $\eta = 0$, then

$$\liminf_{t\to\infty} X_t = 0, \quad \textit{almost surely.}$$

iii) If Condition (B2) holds, if there exists $\varepsilon > 0$ such that

$$\int_0^\infty z\ln^{1+\varepsilon}(1+z)\pi(dz) < \infty$$

and if the function $x \mapsto (\sigma^2(x) + p(x))/x$ is bounded, then

$$\mathbb{P}_x\left(\liminf_{t\to\infty} X_t > 0\right) > 0.$$

In the last case, we additionally prove in the next corollary that with positive probability, X grows (at least) exponentially. Moreover, when the diffusion term is large enough ($\sigma(x)$ larger than $\sqrt{x}$, which corresponds to Feller diffusion), we are able to provide a bound on the absorption rate in the two first cases.

Corollary 2.7. *Suppose that Assumptions C and E are satisfied.*

i) If Condition (B1) holds for $\eta > 0$, and $\inf_{x\geq 0} \sigma^2(x)/x > 0$ then
   - *If $\mathbb{E}\left[(\Theta - 1)\ln\Theta\right] < \eta$, there exists $c_1 > 0$ such that for any $x > 0$*

$$\mathbb{P}_x(X_t > 0) \leq c_1 x e^{\underline{r}(\mathbb{E}[\ln 1/\Theta]-\eta-1/2)t}, \quad (t\to\infty).$$

   - *If $\mathbb{E}\left[(\Theta - 1)\ln\Theta\right] = \eta$, there exists $c_2 > 0$ such that for any $x > 0$*

$$\mathbb{P}_x(X_t > 0) \leq c_2 x t^{-1/2} e^{\underline{r}(\mathbb{E}[\ln 1/\Theta]-\eta-1/2)t}, \quad (t\to\infty).$$

– *If* $\mathbb{E}\left[(\Theta-1)\ln\Theta\right]>\eta$*, there exists* $c_3(x)>0$ *such that for any* $x>0$

$$\mathbb{P}_x(X_t>0)\leq c_3(x)t^{-3/2}e^{\underline{r}(\mathbb{E}[\ln 1/\Theta]-\eta+\mathbb{E}[(\Theta^{\tau}-1)])t},\quad (t\to\infty),$$

where $\tau\in[0,1)$ *is the unique value such that* $\mathbb{E}[\ln 1/\Theta]-\eta+\mathbb{E}\left[\Theta^{\tau}\ln\Theta\right]=0$.

ii) If Condition (B1) holds for $\eta=0$*, and* $\inf_{x\geq 0}\sigma^2(x)/x>0$ *then, there exists* $c_4(x)>0$ *such that for any* $x>0$

$$\mathbb{P}_x(X_t>0)\leq c_4(x)t^{-1/2},\quad (t\to\infty).$$

iii) Under the assumptions of point iii) of Proposition 2.6, there exists a Lévy process Λ_t *with drift* η *and an increasing function* ρ *such that* $\rho(t)\geq\underline{r}t$ *and*

$$\liminf_{t\to\infty}X_t e^{-\Lambda_{\rho(t)}}=W \tag{2.14}$$

where W *is a finite nonnegative random variable satisfying* $\mathbb{P}(W>0)>0$.

Absorption rates of CSBPs in random environment have been intensively studied during the last decade [9, 3, 32, 27, 4]. In these references, g is constant, $\sigma^2(x)=\sigma^2x$, for some $\sigma\geq 0$, $p(x)=x$ and $r(x)\equiv r$ is independent of X, whereas these assumptions are relaxed in our case (notice however that we make moment assumptions on the jump measures). Corollary 2.7 provides thus bounds on the asymptotic behaviour for a new class of models.

3. Behaviour of the process at the population level

We now investigate the behaviour of the Markov process at the population level, and introduce an auxiliary process providing information on the behaviour of a "typical individual". To ease the statement of the results, we consider the case of the modelling of a parasite infection in a cell population.

3.1. Construction of the auxiliary process and stochastic differential equation.

Recall from (1.2) that the population state at time t, Z_t, can be represented by a sum of Dirac masses. We denote by $(M_t,t\geq 0)$ the first-moment semi-group associated with the population process Z given for all measurable function f and $x,t\geq 0$ by

$$M_tf(x)=\mathbb{E}_{\delta_x}\left[\sum_{u\in V_t}f(X_t^u)\right].$$

The trait of a typical individual in the population is characterized by the so-called auxiliary process Y (see [28, Theorem 3.1] for detailed computations and proofs) corresponding to the following time-inhomogeneous semi-group for $r\leq s\leq t$, $x\geq 0$:

$$P_{r,s}^{(t)}f(x)=\frac{M_{s-r}(fM_{t-s}\mathbf{1})(x)}{M_{t-r}\mathbf{1}(x)},$$

where $\mathbf{1}$ is the constant function on $\mathbb{R}_+$ equal to 1. More precisely, if we denote by $m(x,s,t)=M_{t-s}\mathbf{1}(x)$ the mean number of individuals in the population at time t starting from one individual with trait x at time s with $s\leq t$, then, for all measurable bounded function $F:\mathbb{D}([0,t],\mathbb{R}_+)\to\mathbb{R}$, we have:

$$\mathbb{E}_{\delta_x}\left[\sum_{u\in V_t}F\left(X_s^u,s\leq t\right)\right]=m(x,0,t)\mathbb{E}_x\left[F\left(Y_s^{(t)},s\leq t\right)\right]. \tag{3.1}$$

Here $(Y_s^{(t)},s\leq t)$ is a time-inhomogeneous Markov process whose law is characterized by its associated infinitesimal generator $(\mathcal{A}_s^{(t)},s\leq t)$ given for $f\in\mathcal{D}(\mathcal{A})$ and $x\geq 0$ by:

$$\mathcal{A}_s^{(t)}f(x)=\widehat{\mathcal{G}}_s^{(t)}f(x)+\widehat{r}_s^{(t)}(x)\int_{\mathcal{X}}\left(f\left(\theta x\right)-f\left(x\right)\right)\widehat{\kappa}_s^{(t)}\left(x,d\theta\right),$$

where

$$\mathcal{D}(\mathcal{A}) = \left\{ f \in \mathcal{C}_b^2(\mathbb{R}_+) \text{ s.t. } m(\cdot, s, t)f \in \mathcal{C}_b^2(\mathbb{R}_+),\ \forall t \geq 0,\ \forall s \leq t \right\}$$

and

$$\widehat{\mathcal{G}}_s^{(t)} f(x) = \left(g(x)x + 2\sigma^2(x) \frac{\partial_x m(x,s,t)}{m(x,s,t)} + p(x) \int_{\mathbb{R}_+} z \left(\frac{m(x+z,s,t) - m(x,s,t)}{m(x,s,t)} \right) \pi(dz) \right) f'(x)$$
$$+ \sigma^2(x) f''(x) + p(x) \int_{\mathbb{R}_+} (f(x+z) - f(x) - zf'(x)) \frac{m(x+z,s,t)}{m(x,s,t)} \pi(dz),$$

$$\widehat{r}_s^{(t)}(x) = 2r(x) \int_0^1 \frac{m(\theta x, s, t)}{m(x,s,t)} \kappa(d\theta),$$

$$\widehat{\kappa}_s^{(t)}(x, d\theta) = \mathbf{1}_{\{0 \leq \theta \leq 1\}} m(\theta x, s, t) \left(\int_0^1 m(\theta x, s, t) \kappa(d\theta) \right)^{-1} \kappa(d\theta).$$

Explicit expressions for the mean number of individuals are usually out of range. However, the computations are duable in two particular cases. In the first one, the parasites Malthusian growth is constant and the cell division rate is a linear function of the quantity of parasites, as stated in the next assumption.

Assumption F. *There exist $\alpha > 0$, $g, \beta \geq 0$ such that $g(x) \equiv g$, $r(x) = \alpha x + \beta$, and*

$$\int_0^\infty z^2 \pi(dz) < \infty. \tag{3.2}$$

Such a division rate corresponds for the cell to a strategy of linear increase of its division rate in order to get rid of the parasites. Under Assumption F, a direct computation shows that if $g \neq \beta$, the mean number of individuals can be written

$$m(x,s,t) = \frac{\alpha x}{g - \beta} e^{g(t-s)} + \left(1 - \frac{\alpha x}{g - \beta} \right) e^{\beta(t-s)}. \tag{3.3}$$

As a consequence, we obtain the following expressions for $\widehat{\mathcal{G}}$, $\widehat{r}$ and $\widehat{\kappa}$:

$$\widehat{\mathcal{G}}_s^{(t)} f(x) = \left(gx + \left(2\sigma^2(x) + p(x)\mathbb{E}\left[\mathcal{Z}^2\right]\right) \frac{\alpha \left(e^{(g-\beta)(t-s)} - 1\right)}{(g-\beta) + \alpha x \left(e^{(g-\beta)(t-s)} - 1\right)} \right) f'(x) + \sigma^2(x) f''(x)$$
$$+ p(x) \int_{\mathbb{R}_+} (f(x+z) - f(x) - zf'(x)) \left(1 + \frac{\alpha z \left(e^{(g-\beta)(t-s)} - 1\right)}{(g-\beta) + \alpha x \left(e^{(g-\beta)(t-s)} - 1\right)} \right) \pi(dz),$$

$$\widehat{r}_s^{(t)}(x) = (\alpha x + \beta) \left(1 + \frac{g - \beta}{g - \beta + \alpha x \left(e^{(g-\beta)(t-s)} - 1\right)} \right),$$

and

$$\widehat{\kappa}_s^{(t)}(x, d\theta) = \mathbf{1}_{\{0 \leq \theta \leq 1\}} 2 \frac{g - \beta + \alpha \theta x \left(e^{(g-\beta)(t-s)} - 1\right)}{2(g-\beta) + \alpha x \left(e^{(g-\beta)(t-s)} - 1\right)} \kappa(d\theta),$$

where $\mathcal{Z}$ is a positive random variable with law π. For the sake of readability, we introduce the following functions for $y > 0$, $s, z \geq 0$, and $\theta \in [0,1]$:

$$f_1(y,s) := g + \left(2\frac{\sigma^2(y)}{y} + \mathbb{E}\left[\mathcal{Z}^2\right] \frac{p(y)}{y} \right) \frac{\alpha \left(e^{(g-\beta)s} - 1\right)}{g - \beta + \alpha y \left(e^{(g-\beta)s} - 1\right)},$$

$$f_2(y,s,\theta) := 2(\alpha y+\beta)\frac{g-\beta+\alpha\theta y\left(e^{(g-\beta)s}-1\right)}{g-\beta+\alpha y\left(e^{(g-\beta)s}-1\right)},$$

and

$$f_3(y,s,z) := p(y)\left(1+\frac{\alpha z\left(e^{(g-\beta)s}-1\right)}{(g-\beta)+\alpha y\left(e^{(g-\beta)s}-1\right)}\right).$$

We thus obtain that $\mathcal{A}^{(t)}$ is the infinitesimal generator of the solution to the following SDE, when existence and unicity in law of the solution hold: for $0\le s\le t$,

$$\begin{aligned}Y_s^{(t)} =&Y_0^{(t)} + \int_0^s \left[Y_u^{(t)} f_1(Y_u^{(t)},t-u)\right]du + \int_0^s\int_0^\infty\int_0^{f_3(Y_{u^-}^{(t)},t-u,z)} z\widetilde{Q}(du,dz,dx)\\ &+\int_0^s\sqrt{2\sigma^2\left(Y_u^{(t)}\right)}dB_u + \int_0^s\int_0^{f_2(Y_{u^-}^{(t)},t-u,\theta)}\int_0^1(\theta-1)Y_{u^-}^{(t)}N(du,dz,d\theta), \end{aligned}\tag{3.4}$$

where $\widetilde{Q}$, B and N are the same as in (2.1).

For the sake of completeness, we give in the appendix the expressions of the kernels and functions $(f_i, 1\le i\le 3)$ in the case $g=\beta$, as well as the proof of Proposition 3.1 in this case. We will not consider this special case in the sequel, as it entails additional computations and does not bring new insights.

Before studying the auxiliary process, we prove that it can be realised as the unique strong solution to the SDE (3.4) under some moment conditions on the measure associated with positive jumps. We need to consider an additional assumption on p that ensures that the rate of positive jumps f_3 of the process Y is increasing with the quantity of parasites.

Assumption G. *For every $x\ge 0$,*

$$xp'(x)\ge p(x).$$

Notice that Assumption G is satisfied for any function p of the form $p(x)=\chi x^\psi \ln^\omega x$ with $\chi\ge 0$, $\psi\ge 1$, and $\omega\ge 0$.

Proposition 3.1. *Suppose that Assumptions C and G, as well as condition* (3.2) *hold. Then, Equation* (3.4) *has a pathwise unique nonnegative strong solution.*

Remark 3.2. *Notice that the reproduction law could be generalised to allow for a random number of offspring (independent of the individual trait). In this case, the constant* 2 *in $\widehat{r}_s^{(t)}$ would be replaced by the mean number of offspring (see* [2, 28]*).*

The aim of the next sections is to derive properties on the behaviour for large t of the auxiliary process $(Y_s^{(t)}, s\le t)$.

3.2. **Linear division rate: Absorption probability and convergence of the auxiliary process.** In what follows, we set $Y_s^{(t)}=Y_t^{(t)}$ for all $s\ge t$. As for the study of the process along a lineage, we proceed by couplings of random processes several times in the proofs of results stated in Sections 3.2 and 3.3.

The next proposition is the analogue of Corollary 2.4 on the absorption of the auxiliary process in finite time and its proof is very similar, except that we have to deal with time dependencies. Let

$$\tau_t^-(0) := \inf\{0<s\le t : Y_s^{(t)}=0\},$$

with the convention $\inf\emptyset := \infty$.

Proposition 3.3. *Suppose that Assumptions C, E, F and G hold. Then points i) and iii) of Corollary 2.4 hold with $\tau_t^-(0)$ in place of $\tau^-(0)$.*

Notice that in the current case, the assumptions of points *ii)* and *iii)* of Corollary 2.4 are the same, because $r > 0$ on $\mathbb{R}_+^*$.

In the case where the absorption of the auxiliary process occurs with positive probability, we are able to prove the convergence of the last part of the auxiliary process trajectory on a time window of any size.

Proposition 3.4. *Let $T \geq 0$. Suppose that Assumptions C, E, F and G and Condition (A2) hold. Assume further that*

$$\limsup_{x\to\infty}\left(\frac{6\sigma^2(x)+5\mathbb{E}\left[\mathcal{Z}^2\right]p(x)}{x^3}\right) < \frac{2\alpha}{5}\mathbb{E}[\Theta(1-\Theta^5)] \tag{3.5}$$

and that $\mathbb{E}\left[\mathcal{Z}^6\right] < \infty$. Then, there exist $C, \overline{c} > 0$ and a probability measure Π on the Borel σ-field of $\mathbb{D}([0,T],\mathcal{X})$ endowed with the Skorokhod distance such that for all bounded measurable function $F : \mathbb{D}([0,T],\mathcal{X}) \to \mathbb{R}$ and for all $x \geq 0$,

$$\left|\mathbb{E}\left[F\left(Y_{t+s}^{(t+T)}, s\leq T\right)\middle| Y_0^{(t+T)} = x\right] - \Pi(F)\right| \leq Ce^{-\overline{c}t}\left\|F\right\|_\infty x.$$

This convergence result allows us to establish a law of large numbers, linking asymptotically the behaviour of a typical individual, given by the auxiliary process $Y^{(t)}$, with the behaviour of the whole population. To use the results of [29], we need to consider an additional condition on the division rate. This condition is needed to get Many-to-One formulas for the whole tree and forks (see [28, Proposition 3.5, Proposition 3.6]), which are used to control the fluctuations of the population trajectories in the proof of [29, Corollary 3.4].

Assumption H. *For all $x \geq 0$,*

$$\lim_{t\to\infty}\int_0^t r(X_s)ds = \infty, \text{ almost surely.}$$

This assumption is satisfied for instance if $\beta > 0$. Note that in [29], this assumption is also required for the proof of the convergence of the auxiliary process but a close look at the proof shows that it is in fact not needed.

Theorem 3.5. *Under the assumptions of Proposition 3.4 and Assumptions B and H, for all bounded measurable function $F : \mathbb{D}([0,T],\mathcal{X}) \to \mathbb{R}$, for all $x_0, x_1 \geq 0$, we have,*

$$\frac{\sum_{u\in V_{t+T}} F\left(X_{t+s}^u, s\leq T\right)}{N_{t+T}} - \mathbb{E}\left[F\left(Y_{t+s}^{(t+T)}, s\leq T\right)\middle| Y_0^{(t+T)} = x_1\right] \xrightarrow[t\to+\infty]{} 0 \text{ in } \mathbb{L}_2(\delta_{x_0}),$$

where $N_t := \text{Card}(V_t)$.

This result from [29] ensures that asymptotically, the trajectory of the traits of a sampling along its ancestral lineage corresponds to the trajectory of the auxiliary process. Hence, the study of the asymptotic behaviour of the proportion of individuals satisfying some properties, such as the proportion of infected individuals, is reduced to the study of the time-inhomogenous process Y. Notice that unfortunately, the conditions on p for Theorem 3.5 to hold are restrictive, as p has to satisfy

$$\limsup_{x\to 0^+}\frac{p(x)}{x} < \infty, \quad \limsup_{x\to\infty}\frac{p(x)}{x^3} < \infty \quad \text{and} \quad xp'(x) \geq p(x).$$

However, these conditions still cover a class of functions not studied in this context until now (in particular $\chi x^\psi \ln^\omega x$, with $\chi \geq 0$, $1 < \psi \leq 3$, $\omega \geq 0$ ($\omega = 0$ if $\psi = 3$)), and include in particular the classical rate of positive jumps of CSBP (χx, $\chi \geq 0$). Moreover (see Proposition 3.6) we can obtain results on moderate infection without Theorem 3.5. In these cases the assumptions on p are much weaker.

3.3. **Linear division rate: Moderate infection.** We show that if the individual division rate depends linearly on the individual's trait, the infection stays moderate, and the population may even recover under some assumptions, as stated in the next result.

Proposition 3.6. *Suppose that Assumptions B, C, E, F, G and H hold, that $g < \beta$ and that condition* (3.5) *holds.*

i) If either $\limsup_{0+} \sigma^2(y)/y < \infty$ *or* $\mathbb{E}\left[\mathcal{Z}^6\right] < \infty$ *and (A2) holds, then*

$$\lim_{K\to\infty}\lim_{t\to\infty}\mathbb{P}\left(\frac{\sum_{u\in V_t}\mathbf{1}_{\{X_t^u>K\}}}{N_t} > \varepsilon\right) = 0.$$

ii) If $\limsup_{0+} \sigma^2(y)/y < \infty$, $\mathbb{E}\left[\mathcal{Z}^6\right] < \infty$, *and (A2) holds, then*

$$\frac{\sum_{u\in V_t}\mathbf{1}_{\{X_t^u=0\}}}{N_t} \to 1 \quad \textit{in} \quad \mathbb{L}_2(\delta_{x_0}), \quad (t\to\infty).$$

Notice that point *ii)* covers the classical diffusive function ($\sigma^2(x) = \sigma^2 x$, $\sigma > 0$).

Proposition 3.6 extends the results of [6] to a class of division rates increasing with the quantity of parasites. It is similar in spirit to Conjecture 5.1 in [6] in the case of birth rates increasing with the quantity of parasites, but Bansaye and Tran considered a case where the division rate is bounded, which is not our case. Moreover, we consider positive jumps and various diffusive functions for the growth of the parasites.

3.4. **Constant division rate: Long time behaviour of the infection.** We now consider the special case of a constant division rate for cells ($r(x) \equiv \beta > 0$) but keep the possibility for the individual parasites' growth rate g to depend on the quantity of parasites. In this case we can observe moderate infections, but also cases where the quantity of parasites goes to infinity with an exponential growth in a positive fraction of the cells.

When the function r is constant, the auxiliary process $Y^{(t)}$ follows a time homogeneous SDE. Therefore, in this section, we drop the dependence in t and we denote by Y the auxiliary process. More precisely, applying the results of Section 3.1 we obtain that Y is the unique strong solution to the following SDE. For all $s \geq 0$,

$$Y_s = Y_0 + \int_0^s g(Y_u)Y_u du + \int_0^s \sqrt{2\sigma^2(Y_u)}dB_u + \int_0^s\int_0^{p(Y_{u^-})}\int_{\mathbb{R}_+} z\widetilde{Q}(du,dx,dz) + \int_0^s\int_0^{2\beta}\int_0^1 (\theta-1)Y_{u^-}N(du,dz,d\theta).$$

In other words, the auxiliary process has the same law as the process along a lineage with a cell division rate multiplied by two (see [2, 6]).

Proposition 3.7. *Suppose that Assumptions B, C, E hold, and that* $r(x) \equiv \beta > 0$.

i) If there exists $\eta > 0$ *such that for* $x \geq 0$, $g(x) + 2\beta\mathbb{E}[\ln\Theta] > \eta$, *the function* $x \mapsto (\sigma^2(x) + p(x))/x$ *is bounded and there exists* $\varepsilon_1 > 0$ *such that*

$$\int_{\mathbb{R}_+} z\ln^{1+\varepsilon_1}(1+z)\pi(dz) < \infty,$$

then for $\varepsilon > 0$

$$\liminf_{t\to\infty}\mathbb{E}\left[\frac{\sum_{u\in V_t}\mathbf{1}_{\{X_t^u>e^{(\eta-\varepsilon)t}\}}}{N_t}\right] > 0 \quad \textit{and} \quad \left\{\limsup_{t\to\infty}\frac{\sum_{u\in V_t}\mathbf{1}_{\{X_t^u>e^{(\eta-\varepsilon)t}\}}}{N_t}\right\} = \left\{Y_t^{(t)} > 0, \forall t\geq 0\right\}.$$

ii) If there exists $\eta > 0$ *such that for* $x \geq 0$, $g(x) + 2\beta\mathbb{E}[\ln\Theta] < -\eta$,

$$\frac{\sum_{u\in V_t}\mathbf{1}_{\{X_t^u>\varepsilon\}}}{N_t} \to 0 \quad \textit{in probability}, \quad (t\to\infty).$$

iii) If Assumption (A2) holds and if there exist $\eta > 0$ and $x_0 \geq 0$ such that for $x \geq x_0$, $g(x) + 2\beta\mathbb{E}[\ln \Theta] < -\eta$, then

$$\frac{\sum_{u\in V_t} \mathbf{1}_{\{X_t^u=0\}}}{N_t} \to 1 \quad a.s., \quad (t\to\infty).$$

Proposition 3.7 extends Theorem 4.2 in [6] allowing for non constant parasite's growth rates, a general class of diffusive functions, and positive jumps.

4. Proofs

4.1. Proofs of Section 2.1.

Proof of Proposition 2.1. The proof is a direct application of Proposition 1 in [30]. First according to their conditions (i) to (iv) on page 60, our parameters are admissible. Second, we need to check that conditions (a), (b) and (c) in [30] are fulfilled.

In our case, condition (a) writes as follows: for any $n \in \mathbb{N}$, there exists $A_n < \infty$ such that for any $0 \leq x \leq n$,

$$\int_0^\infty \int_0^1 \left|(\theta-1)x\mathbf{1}_{\{z\leq r(x)\}}\right| dz\kappa(d\theta) \leq A_n(1+x).$$

But we have the equality

$$\int_0^\infty \int_0^1 \left|(\theta-1)x\mathbf{1}_{\{z\leq r(x)\}}\right| dz\kappa(d\theta) = \frac{1}{2}xr(x),$$

and thus (a) holds, as r is continuous.

To satisfy condition (b), it is enough to check that for any $n\in\mathbb{N}$ there exists $B_n < \infty$ such that for $0 \leq x \leq y \leq n$,

$$|xg(x)-yg(y)|+\int_0^\infty \int_0^1 (1-\theta)\left|x\mathbf{1}_{\{u\leq r(x)\}} - y\mathbf{1}_{\{u\leq r(y)\}}\right| du\kappa(d\theta) \leq B_n(y-x)\left(\ln\left((y-x)^{-1}\right)+1\right).$$

Indeed, the function $\phi : x \mapsto x(\ln(1/x)+1)$ is concave and non-decreasing on $[0,1/e]$, and satisfies $\int_{0+} 1/\phi = \infty$. Now we have the following series of equalities:

$$\begin{aligned}\int_0^\infty \left|x\mathbf{1}_{\{u\leq r(x)\}} - y\mathbf{1}_{\{u\leq r(y)\}}\right| du = \\ \int_0^\infty \left((y-x)\mathbf{1}_{\{u\leq (r(x)\wedge r(y))\}} + y\mathbf{1}_{\{r(x)<u\leq r(y)\}} + x\mathbf{1}_{\{r(y)<u\leq r(x)\}}\right) du \\ = \mathbf{1}_{\{r(x)<r(y)\}}(yr(y)-xr(x)) + \mathbf{1}_{\{r(y)\leq r(x)\}}(yr(y)+xr(x)-2xr(y)).\end{aligned}$$

This yields that condition (b) holds under Assumption C.

Finally, let us focus on (c), which can be rewritten as follows: for any $n\in\mathbb{N}$ there exists $D_n < \infty$ such that for $0 \leq x, y \leq n$,

$$|\sigma(x)-\sigma(y)|^2+\int_0^\infty \int_0^\infty \left(\left|\mathbf{1}_{u\leq p(x)}z - \mathbf{1}_{u\leq p(y)}z\right| \wedge \left|\mathbf{1}_{u\leq p(x)}z - \mathbf{1}_{u\leq p(y)}z\right|^2\right)\pi(dz)du \leq D_n|x-y|.$$

The first term fulfills the condition as σ is 1/2-Hölderian by Assumption A. The second term is equal to

$$\int_0^\infty (z\wedge z^2)\pi(dz)\int_0^\infty \left|\mathbf{1}_{u\leq p(x)} - \mathbf{1}_{u\leq p(y)}\right| du = \left(\int_0^\infty (z\wedge z^2)\pi(dz)\right)|p(x)-p(y)|.$$

This ends the proof, according to Assumption A. □

4.2. **Proofs of Section 2.2.** We first prove Theorem 2.3. As mentioned previously, the proof is close to the one of [26, Theorem 2.3]. However, as we extend this theorem, several steps of the proof are modified. For the sake of readability we provide the whole proof, including parts which were done similarly in [26].

Proof of Theorem 2.3. We first focus on point *i)*. Let $n \in \mathbb{N}$ be such that $n \geq 2$ and let $0 < \varepsilon < b$ be such that $\varepsilon < 1$ and (2.4) holds for all $u \leq b$. Let $T_n = \tau^-(\varepsilon^n) \wedge \tau^+(b)$. By assumption, we have for $a > 1$

$$\varepsilon^{1-a} \geq \mathbb{E}_\varepsilon \left[X_{T_n}^{1-a} \exp\left(\int_0^{T_n} G_a\left(X_s\right) ds \right) \right] \geq \mathbb{E}_\varepsilon \left[X_{T_n}^{1-a} \exp\left(-T_n Z(\Theta, \varepsilon^n)\right) \mathbf{1}_{\{\tau^-(\varepsilon^n) < \tau^+(b)\}} \right], \tag{4.1}$$

where

$$Z(\Theta, \varepsilon^n) = \ln((\Theta\varepsilon^n)^{-1})(f((\Theta\varepsilon^n)^{-1}))^{-2}$$

and Θ is a random variable distributed according to κ and independent of $X_{T_n^-}$. This bound comes from the fact that $x \mapsto \ln(1/x) f(1/x)^{-2}$ is a decreasing function and it also holds if X reaches ε^n without jumping. Let

$$d_n := \frac{-\ln\left(\varepsilon^{(a-1)n/2}\right)}{\mathbb{E}\left[Z(\Theta, \varepsilon^n)\right]} \geq f\left((\Theta\varepsilon^n)^{-1}\right)^2 \frac{(a-1)/2\left(n \ln \varepsilon^{-1}\right)}{\mathbb{E}\left[\ln(\Theta^{-1})\right] + n \ln \varepsilon^{-1}} \to \infty, \quad (n \to \infty).$$

As $a > 1$, we have $X_{T_n}^{1-a} \mathbf{1}_{\{\tau^-(\varepsilon^n) < \tau^+(b)\}} \geq (\varepsilon^n)^{1-a} \mathbf{1}_{\{\tau^-(\varepsilon^n) < \tau^+(b)\}}$. Then, we get from (4.1),

$$\begin{aligned}
\varepsilon^{1-a} &\geq (\varepsilon^n)^{1-a}\, \mathbb{E}_\varepsilon \left[\exp\left(-d_n Z(\Theta, \varepsilon^n)\right) \mathbf{1}_{\{\tau^-(\varepsilon^n) < \tau^+(b) \wedge d_n\}}\right] \\
&\geq (\varepsilon^n)^{1-a}\, \mathbb{E}\left[\exp\left(\ln\left(\varepsilon^{(a-1)n/2}\right) \frac{Z(\Theta, \varepsilon^n)}{\mathbb{E}\left[Z(\Theta, \varepsilon^n)\right]}\right)\right] \mathbb{P}_\varepsilon\left(\tau^-(\varepsilon^n) < \tau^+(b) \wedge d_n\right),
\end{aligned}$$

because Θ is independent of $X_{T_n^-}$. Using that for all $\alpha, \delta > 0$,

$$\mathbb{E}\left[e^{-\alpha \frac{Z}{\mathbb{E}[Z]}}\right] \geq \mathbb{E}\left[e^{-\alpha \frac{Z}{\mathbb{E}[Z]}} \mathbf{1}_{\{Z \leq \delta \mathbb{E}[Z]\}}\right] \geq e^{-\alpha\delta} \mathbb{P}\left(Z \leq \delta \mathbb{E}[Z]\right) \geq e^{-\alpha\delta}(1 - \delta^{-1})$$

yields for all $\delta > 1$

$$\varepsilon^{1-a} \geq (1 - \delta^{-1}) \varepsilon^{(1-a)n} \varepsilon^{(a-1)n\delta/2} \mathbb{P}_\varepsilon\left(\tau^-(\varepsilon^n) < \tau^+(b) \wedge d_n\right).$$

We thus get that for all $1 < \delta < 2$,

$$\mathbb{P}_\varepsilon\left(\tau^-(\varepsilon^n) < \tau^+(b) \wedge d_n\right) \leq \frac{\delta}{\delta - 1} \varepsilon^{(a-1)((2-\delta)n-2)/2}.$$

By the Borel-Cantelli Lemma, we have

$$\mathbb{P}_\varepsilon\left(\tau^-(\varepsilon^n) < \tau^+(b) \wedge d_n \quad \text{i.o.}\right) = 0, \tag{4.2}$$

where i.o. stands for infinitely often. As a consequence we get that, $\mathbb{P}_\varepsilon$-a.s.,

$$\tau^-(\varepsilon^n) \geq \tau^+(b) \wedge d_n$$

for n large enough. If there are infinitely many n so that

$$\tau^-(\varepsilon^n) \geq d_n, \tag{4.3}$$

then we have $\tau^-(0) = \infty$. If (4.3) holds for at most finitely many n, then by (4.2), we have $\tau^-(\varepsilon^n) > \tau^+(b)$ for all n large enough. We conclude that for all $0 < \varepsilon < b$ such that (2.4) holds for all $u \leq b$,

$$\mathbb{P}_\varepsilon\left(\tau^-(0) = \infty \text{ or } \tau^+(b) < \tau^-(0)\right) = 1. \tag{4.4}$$

We will now use a coupling to show that $\mathbb{P}_\varepsilon(\tau^-(0) < \infty) = 0$. Let for $N \in \mathbb{N}$,

$$r_{[0,N]} := \sup_{0 \leq x \leq N} r(x),$$

(which is finite as r is a continuous function) and $\widetilde{X}$ be the unique strong solution to

$$\widetilde{X}_t = \widetilde{X}_0 + \int_0^t g(\widetilde{X}_s)\widetilde{X}_s ds + \int_0^t \sqrt{2\sigma^2(\widetilde{X}_s)} dB_s + \int_0^t \int_0^{p(\widetilde{X}_{s-})} \int_{\mathbb{R}_+} z\widetilde{Q}(ds, dx, dz)$$
$$+ \int_0^t \int_0^{r_{[0,N]}} \int_0^1 (\theta - 1)\widetilde{X}_s N(ds, dz, d\theta),$$

where the Brownian motion B and the Poisson random measures $\widetilde{Q}$ and N are the same as in (2.1). We will use four properties of this equation.

a) First it has a unique strong solution according to Proposition 2.1.
b) If $\widetilde{X}^{(1)}$ and $\widetilde{X}^{(2)}$ are two solutions with $\widetilde{X}_0^{(1)} \leq \widetilde{X}_0^{(2)}$, then $\widetilde{X}_t^{(1)} \leq \widetilde{X}_t^{(2)}$ for any positive t (because p is a nondecreasing function according to Assumption E0).
c) If $\widetilde{X}$ is a solution with $\widetilde{X}_0 = X_0$, then $\widetilde{X}_t \leq X_t$ for any t smaller than $\tau^-(0) \wedge \tau^+(N)$ (using again Assumption E0).
d) Equation (4.4) holds also for $\widetilde{X}$.

Our aim now is to prove that

$$\mathbb{P}_\varepsilon \left(\widetilde{\tau}^-(0) < \infty \right) = 0, \tag{4.5}$$

where the $\widetilde{\tau}$'s are defined as the τ's in (2.2) and (2.3) but for the process $\tilde{X}$. Using the coupling described in point c), it will imply that

$$\mathbb{P}_\varepsilon \left(\tau^+(N) \leq \tau^-(0) \right) = 1,$$

and letting N tend to infinity, we will get

$$\mathbb{P}_\varepsilon \left(\tau^-(0) = \infty \right) = 1.$$

Before proceeding to the proof of (4.5), let us notice that from coupling b) we have:

$$\mathbb{E}_{\mathfrak{b}} \left[e^{-\lambda \widetilde{\tau}^-(\varepsilon)} \mathbf{1}_{\{\widetilde{\tau}^-(\varepsilon) < \infty\}} \right] \leq \mathbb{E}_b \left[e^{-\lambda \widetilde{\tau}^-(\varepsilon)} \mathbf{1}_{\{\widetilde{\tau}^-(\varepsilon) < \infty\}} \right] \quad \forall\, b \leq \mathfrak{b}. \tag{4.6}$$

Now the strategy to prove (4.5) will be to show that for any $\lambda > 0$

$$\mathfrak{A}(\lambda, \varepsilon) := \int_0^1 \mathbb{E}_{\theta\varepsilon} \left[e^{-\lambda \widetilde{\tau}^-(0)} \mathbf{1}_{\{\widetilde{\tau}^-(0) < \infty\}} \right] \kappa(d\theta) = 0.$$

For any $0 < \theta \leq 1$, (4.4) yields

$$\begin{aligned} \mathbb{E}_{\theta\varepsilon} \left[e^{-\lambda \widetilde{\tau}^-(0)} \mathbf{1}_{\{\widetilde{\tau}^-(0) < \infty\}} \right] &= \mathbb{E}_{\theta\varepsilon} \left[e^{-\lambda \widetilde{\tau}^-(0)} \mathbf{1}_{\{\widetilde{\tau}^+(b) < \widetilde{\tau}^-(0) < \infty\}} \right] \\ &\leq \mathbb{E}_{\theta\varepsilon} \left[e^{-\lambda \widetilde{\tau}^+(b)} \mathbf{1}_{\{\widetilde{\tau}^+(b) < \widetilde{\tau}^-(0)\}} \right] \mathbb{E}_b \left[e^{-\lambda \widetilde{\tau}^-(0)} \mathbf{1}_{\{\widetilde{\tau}^-(0) < \infty\}} \right], \end{aligned}$$

where the last inequality comes from the Markov property combined with (4.6). Moreover, using again the Markov property, we have

$$\mathbb{E}_b \left[e^{-\lambda \widetilde{\tau}^-(0)} \mathbf{1}_{\{\widetilde{\tau}^-(0) < \infty\}} \right] = \mathbb{E}_b \left[e^{-\lambda \widetilde{\tau}^-(\varepsilon)} \mathbf{1}_{\{\widetilde{\tau}^-(\varepsilon) < \infty\}} \mathbb{E}_{X_{\widetilde{\tau}^-(\varepsilon)}} \left[e^{-\lambda \widetilde{\tau}^-(0)} \mathbf{1}_{\{\widetilde{\tau}^-(0) < \infty\}} \right] \right].$$

The process can cross the level ε either because of the diffusion or because of a negative jump. In both cases, $X_{\widetilde{\tau}^-(\varepsilon)} \geq \varepsilon\Theta$ almost surely, where Θ is a random variable distributed according to κ and independent of the process before time $\widetilde{\tau}^-(\varepsilon)$. Then, using again (4.6),

$$\mathbb{E}_b \left[e^{-\lambda \widetilde{\tau}^-(0)} \mathbf{1}_{\{\widetilde{\tau}^-(0) < \infty\}} \right] \leq \mathbb{E}_b \left[e^{-\lambda \widetilde{\tau}^-(\varepsilon)} \mathbf{1}_{\{\widetilde{\tau}^-(\varepsilon) < \infty\}} \right] \int_0^1 \mathbb{E}_{\theta\varepsilon} \left[e^{-\lambda \widetilde{\tau}^-(0)} \mathbf{1}_{\{\widetilde{\tau}^-(0) < \infty\}} \right] \kappa(d\theta).$$

We thus get

$$\mathfrak{A}(\lambda, \varepsilon) \leq \mathbb{E}_b \left[e^{-\lambda \widetilde{\tau}^-(\varepsilon)} \mathbf{1}_{\{\widetilde{\tau}^-(\varepsilon) < \infty\}} \right] \mathfrak{A}(\lambda, \varepsilon).$$

As

$$\mathbb{E}_b\left[e^{-\lambda\widetilde{\tau}^-(\varepsilon)}\mathbf{1}_{\{\widetilde{\tau}^-(\varepsilon)<\infty\}}\right]<1,$$

we conclude that $\mathfrak{A}(\lambda,\varepsilon)=0$, which ends the proof of point *i)*.

Let us now focus on point *ii)*. Let $0<a<1$, $\eta>0$, $\delta<(3-2a)^{-1}$ and $\varepsilon<e^{-(1-\delta)^{-1}}$ such that (4.14) is satisfied. Let $T=\tau^-(\varepsilon^{1+\delta})\wedge\tau^+(\varepsilon^{1-\delta})$. By Assumption D, for all $z>0$ such that $\varepsilon^{1+\delta}<z<\varepsilon^{1-\delta}$,

$$\begin{aligned}z^{1-a}&\geq\mathbb{E}_z\left[\left(X_{\tau^+(\varepsilon^{1-\delta})}\right)^{1-a}\exp\left(\int_0^{\tau^+(\varepsilon^{1-\delta})}G_a\left(X_u\right)du\right)\mathbf{1}_{\{\tau^+(\varepsilon^{1-\delta})<\tau^-(\varepsilon^{1+\delta})\}}\right]\\&\geq\varepsilon^{(1-\delta)(1-a)}\mathbb{P}_z\left(\tau^+(\varepsilon^{1-\delta})<\tau^-(\varepsilon^{1+\delta})\right),\end{aligned}$$

where we have used that if (2.5) holds, then $G_a(z)\geq0$ for z small enough. Therefore,

$$\mathbb{P}_z\left(\tau^+(\varepsilon^{1-\delta})<\tau^-(\varepsilon^{1+\delta})\right)\leq\varepsilon^{(\delta-1)(1-a)}z^{1-a}.\tag{4.7}$$

Similarly, for every $t\geq0$

$$\begin{aligned}z^{1-a}&\geq\mathbb{E}_z\left[\left(X_t\right)^{1-a}\exp\left(\int_0^t G_a\left(X_s\right)ds\right)\mathbf{1}_{\{\tau^+(\varepsilon^{1-\delta})=\tau^-(\varepsilon^{1+\delta})=\infty\}}\right]\\&\geq\varepsilon^{(1+\delta)(1-a)}e^{t\ln(\varepsilon^{-(1-\delta)})\ln(\ln(\varepsilon^{-(1-\delta)}))^{1+\eta}}\mathbb{P}_z\left(\tau^+(\varepsilon^{1-\delta})=\tau^-(\varepsilon^{1+\delta})=\infty\right),\end{aligned}$$

so that

$$\mathbb{P}_z\left(\tau^+(\varepsilon^{1-\delta})=\tau^-(\varepsilon^{1+\delta})=\infty\right)\leq z^{1-a}\varepsilon^{-(1+\delta)(1-a)}\exp\left(-t\ln(\varepsilon^{\delta-1})\ln(\ln(\varepsilon^{-(1-\delta)}))^{1+\eta}\right).$$

Letting t tend to infinity yields

$$\mathbb{P}_z\left(\tau^+(\varepsilon^{1-\delta})=\tau^-(\varepsilon^{1+\delta})=\infty\right)=0.\tag{4.8}$$

Let

$$t(\varepsilon):=\left(\ln(\ln(\varepsilon^{-(1-\delta)}))\right)^{-1-\eta}.\tag{4.9}$$

We have, using (2.5),

$$\begin{aligned}z^{1-a}\geq&\mathbb{E}_z\left[\left(X_{\tau^-(\varepsilon^{1+\delta})}\right)^{1-a}\exp\left(\int_0^{\tau^-(\varepsilon^{1+\delta})}G_a\left(X_u\right)du\right)\mathbf{1}_{\{t(\varepsilon)<\tau^-(\varepsilon^{1+\delta})<\tau^+(\varepsilon^{1-\delta})\}}\right]\\\geq&\exp\left[\ln(\varepsilon^{-(1-\delta)})\left(\ln(\ln(\varepsilon^{-(1-\delta)}))\right)^{1+\eta}t(\varepsilon)\right]\mathbb{E}_z\left[\left(X_{\tau^-(\varepsilon^{1+\delta})}\right)^{1-a}\mathbf{1}_{\{t(\varepsilon)<\tau^-(\varepsilon^{1+\delta})<\tau^+(\varepsilon^{1-\delta})\}}\right]\\=&\varepsilon^{-(1-\delta)}\mathbb{E}_z\left[\left(X_{\tau^-(\varepsilon^{1+\delta})}\right)^{1-a}\mathbf{1}_{\{t(\varepsilon)<\tau^-(\varepsilon^{1+\delta})<\tau^+(\varepsilon^{1-\delta})\}}\right]\\\geq&\varepsilon^{-(1-\delta)}\varepsilon^{(1+\delta)(1-a)}\mathbb{E}[\Theta^{1-a}]\mathbb{P}_z\left(t(\varepsilon)<\tau^-(\varepsilon^{1+\delta})<\tau^+(\varepsilon^{1-\delta})\right),\end{aligned}$$

where we used as before that for all $y\geq0$, $X_{\tau^-(y)}\geq y\Theta$ almost surely where Θ is a random variable distributed according to κ independent of the process before time $\tau^-(y)$. We deduce,

$$\mathbb{P}_z\left(t(\varepsilon)<\tau^-(\varepsilon^{1+\delta})<\tau^+(\varepsilon^{1-\delta})\right)\leq\mathbb{E}[\Theta^{1-a}]^{-1}\varepsilon^{a+(a-2)\delta}z^{1-a}.\tag{4.10}$$

Combining (4.7), (4.8) and (4.10), we get for all $z > 0$ such that $\varepsilon^{1+\delta} < z < \varepsilon^{1-\delta}$,

$$\begin{aligned}\mathbb{P}_z\left(\tau^-(\varepsilon^{1+\delta}) > t(\varepsilon)\right) &\leq \mathbb{E}[\Theta^{1-a}]^{-1}\varepsilon^{a+(a-2)\delta}z^{1-a} + \varepsilon^{(\delta-1)(1-a)}z^{1-a} \\ &= \varepsilon^{(\delta-1)(1-a)}z^{1-a}\left(\mathbb{E}[\Theta^{1-a}]^{-1}\varepsilon^{1-\delta(3-2a)} + 1\right) \\ &\leq \left(\mathbb{E}[\Theta^{1-a}]^{-1} + 1\right)\left(\varepsilon^{(\delta-1)}z\right)^{1-a}, \end{aligned} \tag{4.11}$$

as by assumption δ is smaller than $(3-2a)^{-1}$. By the strong Markov property,

$$\mathbb{P}_z\left(\bigcap_{n=0}^{m}\left\{\tau^-(\varepsilon^{(1+\delta)^n}) < \infty, \tau^-(\varepsilon^{(1+\delta)^{n+1}})\circ\theta_{\tau^-(\varepsilon^{(1+\delta)^n})} \leq t(\varepsilon^{(1+\delta)^n})\right\}\right) \tag{4.12}$$

$$= \mathbb{E}_z\left[\prod_{n=0}^{m}\mathbb{P}_{X_{\tau^-\left(\varepsilon^{(1+\delta)^n}\right)}}\left(\tau^-(\varepsilon^{(1+\delta)^{n+1}}) \leq t(\varepsilon^{(1+\delta)^n})\right)\right],$$

where $\theta_s : \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+) \to \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ is the shift operator $(\theta_s X)(t) = X(s+t)$.

There are two possibilities. If

$$X_{\tau^-(\varepsilon^{(1+\delta)^n})} > \varepsilon^{(1+\delta)^{n+1}},$$

then we can apply (4.11) with $\varepsilon^{(1+\delta)^n}$ instead of ε, and we get

$$\begin{aligned}\mathbb{P}_{X_{\tau^-(\varepsilon^{(1+\delta)^n})}}\left(\tau^-(\varepsilon^{(1+\delta)^{n+1}}) \leq t(\varepsilon^{(1+\delta)^n})\right) &\geq 1 - \left(\mathbb{E}[\Theta^{1-a}]^{-1} + 1\right)\left(\varepsilon^{(\delta-1)(1+\delta)^n}X_{\tau^-(\varepsilon^{(1+\delta)^n})}\right)^{1-a} \\ &\geq 1 - \left(\mathbb{E}[\Theta^{1-a}]^{-1} + 1\right)\left(\varepsilon^{(\delta-1)(1+\delta)^n}\varepsilon^{(1+\delta)^n}\right)^{1-a} \\ &= 1 - \left(\mathbb{E}[\Theta^{1-a}]^{-1} + 1\right)\left(\varepsilon^{\delta(1+\delta)^n}\right)^{1-a}.\end{aligned}$$

Else if

$$X_{\tau^-(\varepsilon^{(1+\delta)^n})} \leq \varepsilon^{(1+\delta)^{n+1}},$$

then

$$\mathbb{P}_{X_{\tau^-(\varepsilon^{(1+\delta)^n})}}\left(\tau^-(\varepsilon^{(1+\delta)^{n+1}}) \leq t(\varepsilon^{(1+\delta)^n})\right) = 1 \geq 1 - \left(\mathbb{E}[\Theta^{1-a}]^{-1} + 1\right)\left(\varepsilon^{\delta(1+\delta)^n}\right)^{1-a}.$$

Combining this inequality with (4.12), we thus obtain

$$\begin{aligned}&\mathbb{P}_z\left(\bigcap_{n=0}^{m}\{\tau^-(\varepsilon^{(1+\delta)^n}) < \infty, \tau^-(\varepsilon^{(1+\delta)^{n+1}})\circ\theta_{\tau^-(\varepsilon^{(1+\delta)^n})} \leq t(\varepsilon^{(1+\delta)^n})\}\right) \\ &\geq \prod_{n=0}^{m}\left(1 - \left(\mathbb{E}[\Theta^{1-a}]^{-1} + 1\right)\varepsilon^{(1-a)\delta(1+\delta)^n}\right) \\ &\geq \prod_{n=0}^{m} e^{-2\left(\mathbb{E}[\Theta^{1-a}]^{-1}+1\right)\varepsilon^{(1-a)\delta(1+\delta)^n}} = e^{-2\left(\mathbb{E}[\Theta^{1-a}]^{-1}+1\right)\sum_{n=0}^{m}\varepsilon^{(1-a)\delta(1+\delta)^n}}\end{aligned} \tag{4.13}$$

where the last inequality holds for

$$\varepsilon < \left(\frac{\ln 2}{2}\right)^{1/\delta(1-a)}\left(\frac{1}{\mathbb{E}[\Theta^{1-a}]^{-1} + 1}\right)^{1/\delta(1-a)}, \tag{4.14}$$

because $x \mapsto 1 - x - e^{-2x}$ is positive for $x \leq \ln(2)/2$. Next,

$$\sum_{n=0}^{m}\varepsilon^{(1-a)\delta(1+\delta)^n} = \varepsilon^{(1-a)\delta}\sum_{n=0}^{m}\varepsilon^{(1-a)\delta((1+\delta)^n-1)} \leq \varepsilon^{(1-a)\delta}\sum_{n=0}^{m}\varepsilon^{(1-a)\delta^2 n} \leq \frac{\varepsilon^{(1-a)\delta}}{1-\varepsilon^{(1-a)\delta^2}}. \tag{4.15}$$

Combining (4.13) and (4.15) and letting $m \to \infty$, we get by monotone convergence

$$\mathbb{P}_z\left(\bigcap_{n=0}^{\infty}\left\{\tau^-(\varepsilon^{(1+\delta)^n}) < \infty, \tau^-(\varepsilon^{(1+\delta)^{n+1}}) \circ \theta_{\tau^-(\varepsilon^{(1+\delta)^n})} \le t(\varepsilon^{(1+\delta)^n})\right\}\right)$$
$$\ge e^{-2\left(\mathbb{E}[\Theta^{1-a}]^{-1}+1\right)\varepsilon^{(1-a)\delta}(1-\varepsilon^{(1-a)\delta^2})^{-1}}.$$

Since under $\mathbb{P}_z$,

$$\tau^-(0) = \sum_{n=0}^{\infty} \tau^-(\varepsilon^{(1+\delta)^{n+1}}) \circ \theta_{\tau^-(\varepsilon^{(1+\delta)^n})},$$

then

$$\mathbb{P}_z\left(\tau^-(0)- \le \sum_{n=0}^{\infty} t(\varepsilon^{(1+\delta)^n})\right) \ge e^{-2\left(\mathbb{E}[\Theta^{1-a}]^{-1}+1\right)\varepsilon^{(1-a)\delta}(1-\varepsilon^{(1-a)\delta^2})^{-1}}.$$

Notice that for $\varepsilon_n := \varepsilon^{(1+\delta)^n}$,

$$t(\varepsilon_n) = \left(\ln(\ln(\varepsilon^{-(1-\delta)(1+\delta)^n}))\right)^{-(1+\eta)} = \left(n\ln(1+\delta) + \ln(1-\delta) + \ln(\ln(\varepsilon^{-1})\right)^{-(1+\eta)}.$$

In particular, for large n,

$$t(\varepsilon_n) \sim (n\ln(1+\delta))^{-(1+\eta)}.$$

This ensures that

$$\sum_{n=1}^{\infty} t(\varepsilon_n) < \infty.$$

We thus have

$$\mathbb{P}_z\left(\tau^-(0)- < \infty\right) \ge e^{-2\left(\mathbb{E}[\Theta^{1-a}]^{-1}+1\right)\varepsilon^{(1-a)\delta}(1-\varepsilon^{(1-a)\delta^2})^{-1}} > 0.$$

This proves point *ii)*. □

We now prove that the functions G_a defined in (2.6) satisfy the conditions of Theorem 2.3. Let us first prove that under Assumption E, the part corresponding to positive jumps (with associated measure π) in the definition of G_a is bounded for small x. Note that by Taylor's formula with integral remainder (see Lemma 2.1 in [26]), we have

$$p(x)\int_{\mathbb{R}_+}\left((zx^{-1}+1)^{1-a} - 1 - (1-a)zx^{-1}\right)\pi(dz)$$
$$= a(a-1)p(x)x^{-2}\int_{\mathbb{R}_+} z^2\left(\int_0^1 (1+zx^{-1}v)^{-1-a}(1-v)dv\right)\pi(dz).$$

To show that this term stays bounded in a neighbourhood of 0, we divide the integral into two parts. First

$$\limsup_{0^+}\left(p(x)x^{-2}\int_0^x z^2\left(\int_0^1 (1+zx^{-1}v)^{-1-a}(1-v)dv\right)\pi(dz)\right)$$
$$\le \limsup_{0^+}\left(p(x)x^{-1}\int_0^x \frac{z}{x} z\pi(dz)\right) \le \left(\limsup_{0^+} p(x)x^{-1}\right)\int_{\mathbb{R}_+} z\pi(dz) < \infty,$$

where we used Assumption E. For the second part, we have

$$\begin{aligned}
&\limsup_{0^+}\left(p(x)x^{-2}\int_x^\infty z^2\left(\int_0^1(1+zx^{-1}v)^{-1-a}(1-v)dv\right)\pi(dz)\right)\\
&\quad\le \limsup_{0^+}\left(p(x)x^{-2}\int_x^\infty z^2\left[\left(\int_0^{x/z}dv\right)+\left(\int_{x/z}^1(1+zx^{-1}v)^{-1-a}dv\right)\right]\pi(dz)\right)\\
&\quad= \limsup_{0^+}\left(p(x)x^{-2}\int_x^\infty z^2\left[\frac{x}{z}-\frac{x}{az}\left[(1+zx^{-1}v)^{-a}\right]_{x/z}^1\right]\pi(dz)\right)\\
&\quad= \limsup_{0^+}\left(p(x)x^{-2}\int_x^\infty z^2\left[\frac{x}{z}+\frac{x}{az}\left[2^{-a}-(1+zx^{-1})^{-a}\right]\right]\pi(dz)\right)\\
&\quad\le\left(\limsup_{0^+}p(x)x^{-1}\right)\int_0^\infty z\left[1+\frac{1}{a}2^{-a}\right]\pi(dz)<\infty.
\end{aligned}$$

This allows us to prove the following Lemma.

Lemma 4.1. *Suppose that Assumption C and E hold. Let $t\ge 0$. For all $b>\varepsilon>c>0$, let $T=\tau^-(c)\wedge\tau^+(b)$. Then, for all $a\in\mathcal{A}$, the process*

$$Z^{(a)}_{t\wedge T}:=(X_{t\wedge T})^{1-a}\exp\left(\int_0^{t\wedge T}G_a(X_s)\,ds\right)$$

is a martingale and

$$\mathbb{E}_\varepsilon\left[X_T^{1-a}\exp\left(\int_0^T G_a(X_s)\,ds\right)\right]\le\varepsilon^{1-a}. \tag{4.16}$$

Proof of Lemma 4.1. We follow the ideas of the proof of [26, Lemma 5.1]. Let $a\in\mathcal{A}$. Applying Itô's formula with jumps (see for instance [19, Theorem 5.1] , we have for all $t\ge0$

$$\begin{aligned}
X_t^{1-a}=&X_0^{1-a}+\int_0^t\left[(1-a)g(X_s)X_s^{1-a}-(1-a)aX_s^{-a-1}\sigma^2(X_s)\right]ds\\
&+\int_0^t\int_{\mathbb{R}_+}p(X_s)((z+X_s)^{1-a}-X_s^{1-a}-(1-a)zX_s^{-a})\pi(dz)ds\\
&+\int_0^t\int_0^{r(X_{s^-})}\int_0^1\left(\theta^{1-a}-1\right)X_{s^-}^{1-a}N(ds,du,d\theta)\\
&+(1-a)\int_0^t X_s^{1-a}\sqrt{2\sigma^2(X_s)}dB_s+\int_0^t\int_0^{p(X_{s^-})}\int_{\mathbb{R}_+}\left[(X_{s^-}+z)^{1-a}-X_{s^-}^{1-a}\right]\widetilde{Q}(ds,d\theta,dz)\\
=&X_0^{1-a}-\int_0^t X_s^{1-a}G_a(X_s)ds+M_t,
\end{aligned}$$

where G_a has been defined in (2.6) and $(M_t,t\ge0)$ is a local martingale. Next, using integration by parts we get

$$Z^{(a)}_{t\wedge T}=X_0^{1-a}+\int_0^{t\wedge T}G_a(X_s)Z_sds+\int_0^{t\wedge T}\exp\left(\int_0^s G_a(X_r)\,dr\right)d\left[X^{1-a}\right]_s,$$

so that $\left(Z^{(a)}_{t\wedge T},t\ge0\right)$ is a local martingale. Similarly to [26], we have

$$\mathbb{E}_\varepsilon\left[\sup_{s\le t}(X_{s\wedge T})^{1-a}\exp\left(\int_0^{s\wedge T}G_a(X_r)\,dr\right)\right]<\infty,$$

using Assumptions C and E, so that from [34] p. 38, $\left(Z^{(a)}_{t\wedge T},t\ge0\right)$ is a martingale. Finally, (4.16) follows by Fatou's lemma as in [26]. □

From Lemma 4.1, we know that the set of functions G_a introduced in (2.6) satisfies Assumption D. Next, notice that Assumption (2.7) implies (2.4) and (2.8) implies (2.5). This proves points *i)* and *ii)* of Corollary 2.4. We now prove point *iii)*.

Proof of Corollary 2.4 iii). Assume that for any positive x, $r(x)+\sigma(x)>0$. Let $x_0 \geq 0$ be such that

$$\mathbb{P}_y(\tau^-(0)<\infty)>0, \quad \forall y \leq x_0.$$

Let $y > x_0$. If $r(y)>0$, there exists $\eta_1>0$ such that r stays positive on $[y-\eta_1, y+\eta_1]$ as it is a continuous function according to Assumption C. Hence, for $\eta_2>0$ small enough, starting from y, the intersection of the following events happen with positive probability during the time interval $[0,\eta_2]$:

- the sum of the positive jumps is smaller than $\eta_1/3$:
$$\int_0^{\eta_2}\int_0^{p(X_{s^-})} zQ(ds,dx,dz) < \eta_1/3,$$
due to Assumption E,
- the integral with respect to the compensator of the point process with positive jumps has an absolute value smaller than $\eta_1/3$:
$$\int_0^{\eta_2} p(X_{s^-})zds\pi(dz) < \eta_1/3,$$
due to Assumption E,
- the continuous variation due to g and σ is smaller than $\eta_1/3$:
$$\int_0^{\eta_2} g(X_s)ds + \sup_{s\leq\eta_2}\left|\int_0^s \sqrt{2\sigma^2(X_s)}dB_s\right| \leq \eta_1/3,$$
- there is a negative jump whose size θ satisfies $\theta \leq x_0/(y+\eta_1)$.

As a consequence,

$$\mathbb{P}_y(X_{\eta_2}\leq x_0)>0,$$

and using the Markov property, we obtain

$$\mathbb{P}_y(\tau^-(0)<\infty)>0.$$

Now assume that $r(y)=0$ but $\sigma(y)>0$. As σ is continuous, if $\sigma(z)>0$ for $z\in[x_0,y]$ then

$$\mathbb{P}_y(X_s\leq x_0)>0, \quad \forall s>0$$

thanks to the diffusion and we end the proof by applying again the Markov property. Else, if σ is only positive on an interval of the form $(x_1,y]$ with $x_0<x_1<y$, then by continuity of r and σ given by Assumption A, $r(x_1)>0$ and we are back to the first case. We thus have proven that

$$\mathbb{P}_y(\tau^-(0)<\infty)>0, \quad \forall y\geq 0, \tag{4.17}$$

as soon as $r+\sigma>0$ on $\mathbb{R}_+^*$.

Now, notice that (4.17) implies that for all $y>0$, there exists $K_y>0$ such that

$$\mathbb{P}_y(\tau^-(0)<K_y)>0.$$

Let us define $K_0=\inf_{y>0}K_y$. To obtain a contradiction, we suppose that $K_0>0$. It implies in particular that for any $y>0$,

$$\mathbb{P}_y(X_{2K_0/3}=0)=0,$$

and as a consequence, for any $y>0$, the Markov property yields

$$\mathbb{P}_y(X_{4K_0/3}=0)=\mathbb{E}_y\left[\mathbb{P}_{X_{2K_0/3}}\left(X_{2K_0/3}=0\right)\right]=0,$$

which is a contradiction. Therefore, $K_0 = 0$ so that for any $s > 0$, we can find $x_{s/2}$ such that

$$\mathbb{P}_{x_{s/2}}(\tau^-(0) < s/2) > 0.$$

As X cannot reach 0 by a jump, we know that there exists $0 < y \leq x_{s/2}$ such that

$$\mathbb{P}_y(\tau^-(0) < s/2) > 0 \quad \text{and} \quad \sigma(y) > 0.$$

Now let $x > 0$. As $r + \sigma > 0$ on $\mathbb{R}_+^*$ and $\sigma(y) > 0$ similar arguments as previously imply

$$\mathbb{P}_x(\exists v < s/2, X_v = y_{s/2}) > 0.$$

The strong Markov property allows us to conclude

$$\mathbb{P}_x(\tau^-(0) < s) > 0.$$

This ends the proof of *iii)*. □

We now turn to the proof of Theorem 2.5. First, we prove that if the division mechanism of the cells is stronger than the growth of the parasites in the sense of (B0) for some $x_0 > 0$, then the stopping times $T_i(x_0)$, defined in (2.10), are finite a.s. for all $i \geq 0$.

Lemma 4.2. *Under Assumptions C and E, and if $x_0 > 0$ is such that (B0) is satisfied for some $\eta > 0$, we have $\mathbb{E}[T_i(x_0)] < \infty$ for all $i \geq 0$.*

Proof of Lemma 4.2. To simplify notations, we write $T_i = T_i(x_0)$. According to the strong Markov property, we only have to prove that $\mathbb{E}(T_1) < \infty$. We consider two cases.
i) If $X_{t_0} \leq x_0$ then $T_1 = t_0$. Hence $\mathbb{E}(T_1 \mathbf{1}_{\{X_{t_0} \leq x_0\}}) \leq t_0 < \infty$.
ii) Assume $X_{t_0} > x_0$. By Itô's formula, we have

$$\begin{aligned}\ln(X_{t\wedge T_1}) =& \ln(X_{t_0}) + \int_{t_0}^{t\wedge T_1} g(X_s)ds - \int_{t_0}^{t\wedge T_1} \frac{\sigma^2(X_s)}{X_s^2} ds + M_{t\wedge T_1} \\ &+ \int_{t_0}^{t\wedge T_1} p(X_s)\left[\ln(X_s+z) - \ln(X_s) - z/X_s\right]\pi(dz)ds + \int_{t_0}^{t\wedge T_1} r(X_s)\mathbb{E}\left[\ln\Theta\right]ds,\end{aligned}$$

where $(M_{s\wedge T_1}, s \geq t_0)$ is a martingale with null expectation independent of $\mathcal{F}_{t_0}$. Notice that by the Mean Value Theorem, $\ln(x+z) - \ln(x) - z/x \leq 0$ for all $x, z > 0$. Therefore, for all $t \geq t_0$

$$\begin{aligned}\ln(X_{t\wedge T_1}) - \ln(X_{t_0}) &\leq \int_{t_0}^{t\wedge T_1} (g(X_s) + r(X_s)\mathbb{E}\left[\ln(\Theta)\right]) + M_{t\wedge T_1} \\ &\leq -\eta\,(t\wedge T_1 - t_0) + M_{t\wedge T_1} \end{aligned} \tag{4.18}$$

using (B0). But at time T_1, X may be equal to x_0 if there is no jump, or equal to $X_{T_1^-}\Theta$ where $X_{T_1^-} \geq x_0$, Θ is distributed according to κ, and is independent of $X_{T_1^-}$ and X_{t_0}. As a consequence,

$$\ln\left(\frac{X_{t\wedge T_1}}{X_{t_0}}\right)\mathbf{1}_{\{X_{t_0}\geq x_0\}} \geq \ln\left(\frac{\Theta x_0}{X_{t_0}}\right)\mathbf{1}_{\{X_{t_0}\geq x_0\}}.$$

Thus, combining this last inequality with (4.18), we get that

$$\mathbb{E}[(T_1\wedge t)\mathbf{1}_{\{X_{t_0}\geq x_0\}}] \leq \frac{1}{\eta}\mathbb{E}\left[\ln\left(\frac{X_{t_0}}{\Theta x_0}\right)\mathbf{1}_{\{X_{t_0}\geq x_0\}}\right] + t_0 < \infty,$$

and we obtain the result letting t tend to infinity. □

Proof of Theorem 2.5. Apart from the proof of Equation (2.13), the proof of of Theorem 2.5 follows directly from Lemma 4.2 and [8, Theorem 7.1.4]. It is very similar to the proof of the second point of [18, Theorem 1] for instance and we refer the reader to this paper for details. Equality (2.12) is obtained by taking expectation in (2.1).

Let us now prove (2.13). Let $x_0 > 0$ such that (B0) is satisfied for some $\eta > 0$ and let $\mathfrak{d} > 0$ be such that $\inf_{0\leq y\leq x_0+\mathfrak{d}} r(y) + \sigma(x) > 0$ for all $x > 0$. In order to control uniformly the extinction probability for any initial state in the interval $[0, x_0 + \mathfrak{d}]$, we first introduce a coupling with a process $(\widetilde{X}_t, t \geq 0)$ defined as follows: $\widetilde{X}$ is the unique strong solution to the SDE

$$\widetilde{X}_t = X_0 + \bar{g}\int_0^t \widetilde{X}_s ds + \int_0^t \sqrt{2\sigma^2(\widetilde{X}_s)}dB_s + \int_0^t\int_0^{p(\widetilde{X}_{s-})}\int_{\mathbb{R}_+} z\widetilde{Q}(ds, dx, dz) + \int_0^t\int_0^{\underline{r}}\int_0^1 (\theta-1)\widetilde{X}_s N(ds, dz, d\theta),$$

where

$$\bar{g} := \sup_{0\leq x\leq x_0+\mathfrak{d}} g(x), \quad \underline{r} := \inf_{0\leq x\leq x_0+\mathfrak{d}} r(x),$$

are finite according to Assumption A, and B, $\widetilde{Q}$ and N are the same as in (2.1). For $y \geq 0$, let us introduce

$$S_y := \inf\{s \geq 0, X_s > y\} \quad \text{and} \quad \widetilde{S}_y := \inf\{s \geq 0, \widetilde{X}_s > y\}.$$

Then, using that p is an increasing function (consequence of Assumption E), for any $t \leq \widetilde{S}_{x_0+\mathfrak{d}}$ we have $X_t \leq \widetilde{X}_t$. We obtain in particular for any $x \leq x_0$ and for any $t \geq 0$

$$\mathbb{P}_x(S_{x_0+\mathfrak{d}} > t) \geq \mathbb{P}_x(\widetilde{S}_{x_0+\mathfrak{d}} > t).$$

Finally, notice that if $x_1 \leq x_2 \leq x_0$, for any $t \geq 0$,

$$\mathbb{P}_{x_2}(\widetilde{S}_{x_0+\mathfrak{d}} > t) \leq \mathbb{P}_{x_1}(\widetilde{S}_{x_0+\mathfrak{d}} > t)$$

because the division rate $\underline{r}$ is independent of the process current value and p is an increasing function. Hence we deduce that for all $t_0 \geq 0$,

$$\inf_{0\leq x\leq x_0} \mathbb{P}_x(X_{t_0} = 0) \geq \mathbb{P}_{x_0}(\widetilde{X}_{t_0} = 0, \widetilde{S}_{x_0+\mathfrak{d}} > t_0). \tag{4.19}$$

We will now prove that for every $t_0 > 0$, there exists $\alpha(t_0) > 0$ such that

$$\mathbb{P}_{x_0}(\widetilde{X}_{t_0} = 0, \widetilde{S}_{x_0+\mathfrak{d}} > t_0) = \alpha(t_0).$$

First, note that we can apply Corollary 2.4 *iii)* because by assumption $\underline{r} + \sigma(x) > 0$ on $\mathbb{R}_+$. Therefore, for any $x > 0$, $t_0 > 0$, $\mathbb{P}_x(\widetilde{X}_{t_0} = 0) > 0$. Next, we have

$$0 < \mathbb{P}_x(\widetilde{X}_{t_0} = 0) = \mathbb{P}_{x_0}(\widetilde{X}_{t_0} = 0, \widetilde{S}_{x_0+\mathfrak{d}} > t_0) + \mathbb{P}_{x_0}(\widetilde{X}_{t_0} = 0, \widetilde{S}_{x_0+\mathfrak{d}} \leq t_0).$$

Suppose, to derive a contradiction, that

$$\{\widetilde{X}_{t_0} = 0 | X_0 = x_0\} = \{\widetilde{X}_{t_0} = 0, \widetilde{S}_{x_0+\mathfrak{d}} \leq t_0 | X_0 = x_0\}.$$

This implies

$$\begin{aligned}\mathbb{P}_{x_0}(\widetilde{X}_{t_0} = 0) &= \mathbb{P}_{x_0}(\widetilde{X}_{t_0} = 0, \widetilde{S}_{x_0+\mathfrak{d}} \leq t_0)\\ &\leq \mathbb{P}_{x_0}(\widetilde{X}_{\widetilde{S}_{x_0+\mathfrak{d}}+t_0} = 0, \widetilde{S}_{x_0+\mathfrak{d}} \leq t_0) \leq \mathbb{P}_{x_0}(\widetilde{S}_{x_0+\mathfrak{d}} \leq t_0)\mathbb{P}_{x_0+\mathfrak{d}}(\widetilde{X}_{t_0} = 0)\end{aligned}$$

using the strong Markov property and that $\mathbb{P}_{\widetilde{X}_{\widetilde{S}_{x_0+\mathfrak{d}}}}(\widetilde{X}_{t_0} = 0) \leq \mathbb{P}_{x_0+\mathfrak{d}}(\widetilde{X}_{t_0} = 0)$. Finally, using again that $\widetilde{X}$ is increasing with its initial value and that its division rate is larger than a positive constant, we get

$$\mathbb{P}_{x_0}(\widetilde{X}_{t_0} = 0) \leq \mathbb{P}_{x_0}(\widetilde{S}_{x_0+\mathfrak{d}} \leq t_0)\mathbb{P}_{x_0}(\widetilde{X}_{t_0} = 0) < \mathbb{P}_{x_0}(\widetilde{X}_{t_0} = 0),$$

which yields a contradiction. Therefore,

$$\mathbb{P}_{x_0}(\widetilde{X}_{t_0} = 0, \widetilde{S}_{x_0+\mathfrak{d}} > t_0) > 0$$

and we deduce from (4.19) that for any $t_0 \geq 0$ there exists $\alpha(t_0) > 0$ such that,

$$\inf_{0 \leq x \leq x_0} \mathbb{P}_x(X_{t_0} = 0) \geq \alpha(t_0). \tag{4.20}$$

By the strong Markov property and (4.20), we get for all $x \geq 0$

$$\mathbb{P}_x\left(X_{T_i+t_0} = 0 \middle| (X_t,\ t \leq T_i),\ T_i < \infty\right) \geq \alpha(t_0).$$

Applying Lemma 4.2 and the strong Markov property, we deduce that for any $x \geq 0$,

$$\mathbb{P}_x\left(X_t > 0, \forall\ t \geq 0\right) \leq \mathbb{P}_x\left(\forall i \geq 0,\ X_{T_i+t_0} > 0\right) = \mathbb{P}_x\left(\forall i \geq 0,\ X_{T_i+t_0} > 0,\ T_i < \infty\right) = 0.$$

This concludes the proof. □

4.3. **Proofs of Section 2.3.**

Proof of Proposition 2.6. Let us introduce the following time change:

$$X_t = Y_{\int_0^t r(X_s)ds}. \tag{4.21}$$

According to Theorem 1.4 in Section 6 in [11], there is a version of X satisfying (4.21) for a process Y that is a solution of the martingale problem with associated generator

$$\begin{aligned}\mathcal{G}_Y f(x) =& \frac{xg(x)}{r(x)} f'(x) + \frac{\sigma^2(x)}{r(x)} f''(x) + \int_0^1 (f(\theta x) - f(x))\kappa(d\theta) \\ &+ \frac{p(x)}{r(x)} \int (f(x+z) - f(x) - zf'(x))\pi(dz),\end{aligned}$$

and is a weak solution to

$$\begin{aligned}Y_t = \int_0^t \frac{Y_s g(Y_s)}{r(Y_s)} ds + \sqrt{\frac{2\sigma^2(Y_s)}{r(Y_s)}} dB_s + \int_0^t \int_0^1 \int_0^1 (\theta - 1) Y_{s^-} N(ds, dz, d\theta) \\ + \int_0^t \int_0^{p(Y_{s^-})/r(Y_{s^-})} \int_{\mathbb{R}_+} z\widetilde{Q}(ds, dx, dz),\end{aligned} \tag{4.22}$$

where we choose on purpose the same Poisson Point measures as in the definition of X in (2.1). In fact, as (4.22) admits a unique strong solution (see the proof of Proposition 2.1), Y is even pathwise unique. Now let us introduce the processes $(K_t, t \geq 0)$ and $(Z_t, t \geq 0)$ via

$$K_t := \int_0^t \frac{g(Y_s)}{r(Y_s)} ds + \int_0^t \int_0^1 \int_0^1 \ln\theta N(ds, dz, d\theta)$$

and

$$Z_t := Y_t e^{-K_t}.$$

Then an application of Itô's Formula with jumps gives

$$Z_t = Y_0 + \int_0^t e^{-K_s} \sqrt{\frac{2\sigma^2(Y_s)}{r(Y_s)}} dB_s + \int_0^t \int_0^{p(Y_{s^-})/r(Y_{s^-})} \int_{\mathbb{R}_+} e^{-K_{s^-}} z\widetilde{Q}(ds, dx, dz).$$

Hence $(Z_t, t \geq 0)$ is a non-negative local martingale. In particular it is a non-negative supermartingale and there exists a finite random variable W such that

$$Y_t e^{-K_t} = W, \quad \text{a.s.}, \quad (t \to \infty). \tag{4.23}$$

Under the assumptions of point *i)*, K is smaller than a Lévy process with drift $-\eta$. As a consequence, e^{-K_t} goes to $+\infty$, and we deduce from (4.23) that Y goes to 0. As by assumption $\int_0^t r(X_s)ds \geq \underline{r}t$, we deduce from the time change (4.21) that X goes to 0.

We turn to the proof of *ii)* and consider the associated assumptions. In this case, K is smaller than an oscillating Lévy process, and we have $\liminf_{t\to\infty} K_t = -\infty$. This implies $\liminf_{t\to\infty} Y_t = 0$. Again, we deduce from the time change (4.21) that $\liminf_{t\to\infty} X_t = 0$.

Let us now prove *iii)* as well as point *iii)* of Corollary 2.7. We use arguments similar to the ones needed to prove Corollary 2 in [3]. As we are in a more general setting, we need to adapt several of these arguments. Most adaptations are obtained by couplings with well-chosen processes.

We denote by M a finite bound of the function $x \mapsto (\sigma^2(x)+p(x))/(xr(x))$. The first step consists in showing that $\mathbb{P}(W > 0|K) > 0$. To this aim, we look for a function $\tilde{v}_t(s, \lambda, K, Y)$, differentiable with respect to the variable s, such that $F(s, Z_s)$ is a martingale conditionally on $K = (K_s, s \geq 0)$, where

$$F(s, x) := \exp\{-x\tilde{v}_t(s, \lambda, K, Y)\}.$$

By an application of Itô's Formula with jumps, similarly as in Equation (2.8) in [3], we obtain that $\tilde{v}_t$ has to satisfy for every $s \leq t$,

$$\frac{\partial}{\partial s}\tilde{v}_t(s, \lambda, K, Y) = e^{K_s}\tilde{\psi}_0\left(\tilde{v}_t(s, \lambda, K, Y)e^{-K_s}, Y_s\right), \quad \tilde{v}_t(t, \lambda, K, Y) = \lambda, \tag{4.24}$$

where

$$\tilde{\psi}_0(\phi, x) = \frac{\sigma^2(x)}{xr(x)}\phi^2 + \frac{p(x)}{xr(x)}\int_0^\infty \left(e^{-\phi z} - 1 + \phi z\right)\pi(dz). \tag{4.25}$$

In particular

$$\mathbb{E}_y\left[e^{-\lambda Z_t}\middle|K\right] = e^{-y\tilde{v}_t(0,\lambda,K,Y)}. \tag{4.26}$$

Let us now introduce a function $v_t(s, \lambda, K)$, differentiable with respect to the variable s, and satisfying.

$$\frac{\partial}{\partial s}v_t(s, \lambda, K) = e^{K_s}\psi_0\left(v_t(s, \lambda, K)e^{-K_s}\right), \quad v_t(t, \lambda, K) = \lambda,$$

where

$$\psi_0(\phi) = M\left(\phi^2 + \int_0^\infty \left(e^{-\phi z} - 1 + \phi z\right)\pi(dz)\right).$$

Then for every $\lambda, x \geq 0$,

$$\tilde{\psi}_0(\lambda, x) \leq \psi_0(\lambda)$$

and as a consequence, for all $s \leq t$, $\lambda > 0$

$$v_t(s, \lambda, K) \leq \tilde{v}_t(s, \lambda, K, Y).$$

Combining this last inequality with (4.26), we obtain that

$$\mathbb{E}_y\left[e^{-\lambda Z_t}\middle|K\right] \leq e^{-yv_t(0,\lambda,K)}.$$

Taking $\lambda = 1$ and letting t go to infinity we get

$$\mathbb{E}_y\left[e^{-W}\middle|K\right] \leq e^{-yv_\infty(0,1,K)} < 1,$$

where the last inequality comes from [3] (see the proof of Corollary 2 on page 7). This allows us to conclude that

$$\mathbb{P}(W > 0|K) > 0.$$

Under the assumptions of point *iii)* K is larger than a Lévy process with drift η and as a consequence, e^{-K_t} goes to 0. From (4.23) and the previous inequality, we deduce that

$$\liminf_{t\to\infty} Y_t = \infty$$

with positive probability. In particular, this implies that

$$\liminf_{t\to\infty} X_t = \liminf_{t\to\infty} Y_{\int_0^t r(X_s)ds} \geq \liminf_{t\to\infty} Y_t > 0$$

with positive probability. As $\int_0^t r(X_s)ds \geq \underline{r}t$, we also obtain (2.14). □

Proof of Corollary 2.7. Point *iii)* has been proven with point *iii)* of Proposition 2.6, we thus focus on points *i)* and *ii)*. The idea of the proof is to compare the survival probability of X with the survival probability of a Feller diffusion with jumps, whose asymptotic behaviour has been studied in [3].

Let us recall the definitions of $\tilde{v}$ and $\tilde{\psi_0}$ in (4.24) and (4.25), respectively. Now, according to (B1), $\inf_{x\geq 0} r(x) > 0$ so that by assumption, $\inf_{x\geq 0} \sigma^2(x)/(xr(x)) > 0$. Therefore, there exists $\mathfrak{a} > 0$ such that for every $\phi, x \geq 0$,

$$\tilde{\psi_0}(\phi, x) \geq \mathfrak{a}\phi^2.$$

Hence, if we introduce $\bar{v}$ as the solution to

$$\frac{\partial}{\partial s}\bar{v}_t(s,\lambda,K) = \mathfrak{a}e^{-K_s}\left(\bar{v}_t(s,\lambda,K)\right)^2, \quad \bar{v}_t(t,\lambda,K) = \lambda, \tag{4.27}$$

we obtain that for all $s \leq t$, $\lambda > 0$,

$$\bar{v}_t(s,\lambda,K) \geq \tilde{v}_t(s,\lambda,K,Y),$$

implying, using (4.26),

$$\mathbb{E}_y\left[e^{-\lambda Z_t}\middle| K\right] \geq e^{-y\bar{v}_t(0,\lambda,K)}.$$

Letting λ go to infinity yields

$$\mathbb{P}_y\left(Y_t = 0|K\right) = \mathbb{P}_y\left(Z_t = 0|K\right) \geq e^{-y\bar{v}_t(0,\infty,K)}.$$

But (4.27) has an explicit solution, and

$$\bar{v}_t(0,\infty,K) = \left(\mathfrak{a}\int_0^t e^{-K_u}du\right)^{-1}.$$

We thus deduce that for any $t \geq 0$,

$$\mathbb{P}_y\left(Y_t > 0\right) \leq 1 - \mathbb{E}\left[e^{-y(\mathfrak{a}\int_0^t e^{-K_u}du)^{-1}}\right].$$

A direct application of [3, Theorem 7] with $F(x) = 1 - e^{-y(\mathfrak{a}x)^{-1}}$ gives the long time behaviour of the right hand side of the previous inequality. Finally,

$$\mathbb{P}_y\left(X_t > 0\right) = \mathbb{P}_y\left(Y_{\int_0^t r(X_s)ds} > 0\right) \leq \mathbb{P}_y\left(Y_{\underline{r}t} > 0\right) \leq 1 - \mathbb{E}\left[e^{-y(\mathfrak{a}\int_0^{\underline{r}t} e^{-K_s}ds)^{-1}}\right].$$

□

4.4. **Proofs of Section 3.1.**

Proof of Proposition 3.1. In this section, we provide the proof of the proposition in the case $g \neq \beta$ and we refer the reader to Appendix A for the proof in the case $g = \beta$. The proof is a direct application of [30, Proposition 1]. Notice that in the statement of [30, Proposition 1], the functions b, g and h do not depend on time, whereas it is the case of our process. However this additional dependence does not bring any modification to the proofs (which are mostly derived in the earlier paper [13]). First according to their conditions (i) to (iv) on page 60, our parameters are admissible. Second, we need to check that conditions (a), (b) and (c) are fulfilled.

In our case, condition (a) writes as follows: for any $n \in \mathbb{N}$, there exists $A_n < \infty$ such that for any $0 \leq x \leq n$,

$$\int_0^1 \int_0^\infty \left|(\theta-1)x\mathbf{1}_{\{z\leq f_2(x,s,\theta)\}}\right| dz\kappa(d\theta) \leq A_n(1+x).$$

We have for any $0 \leq x \leq n$

$$\int_0^1 \int_0^\infty \left|(\theta-1)x\mathbf{1}_{\{z\leq f_2(x,s,\theta)\}}\right| dz\kappa(d\theta) = \int_0^1 2(1-\theta)x(\alpha x+\beta)\kappa(d\theta) \leq n(\alpha n+\beta),$$

and thus (a) holds. Now, to satisfy condition (b), it is enough to check that for any $n \in \mathbb{N}$, there exists $B_n(t) < \infty$ such that for $0 \leq s \leq t$ and $0 \leq x \leq y \leq n$,

$$|xf_1(x,s) - yf_1(y,s)| + \int_0^\infty \int_0^1 (1-\theta)\left|x\mathbf{1}_{\{u\leq f_2(x,s)\}} - y\mathbf{1}_{\{u\leq f_2(y,s)\}}\right| du\kappa(d\theta) \leq B_n(t)|y-x|.$$

First, we have

$$xf_1(x,s) = gx + F(x)\frac{\alpha\left(e^{(g-\beta)s}-1\right)}{g-\beta+\alpha x\left(e^{(g-\beta)s}-1\right)},$$

where $F(x) := 2\sigma^2(x) + \mathbb{E}\left[\mathcal{Z}^2\right]p(x)$. For $0 \leq x, y \leq n$,

$$\begin{aligned}
&\left|\frac{F(x)}{g-\beta+\alpha x\left(e^{(g-\beta)s}-1\right)} - \frac{F(y)}{g-\beta+\alpha y\left(e^{(g-\beta)s}-1\right)}\right| \\
&\qquad \leq \left|\frac{(F(x)-F(y))(g-\beta) + \alpha\left(e^{(g-\beta)s}-1\right)(F(x)y-F(y)x)}{\left(g-\beta+\alpha y\left(e^{(g-\beta)s}-1\right)\right)\left(g-\beta+\alpha x\left(e^{(g-\beta)s}-1\right)\right)}\right| \\
&\qquad \leq 2\frac{|\sigma(x)-\sigma(y)|^2}{|g-\beta|} + \mathbb{E}\left[\mathcal{Z}^2\right]\frac{|p(x)-p(y)|}{|g-\beta|} + \left|\frac{\alpha\left(e^{(g-\beta)s}-1\right)}{(g-\beta)^2}\right||F(x)y-F(y)x| \\
&\leq 2\frac{|\sigma(x)-\sigma(y)|^2}{|g-\beta|} + \mathbb{E}\left[\mathcal{Z}^2\right]\frac{|p(x)-p(y)|}{|g-\beta|} + \frac{\alpha\left(e^{(g-\beta)t}+1\right)}{(g-\beta)^2}(F(x)|y-x| + |F(x)-F(y)|\,x).
\end{aligned}$$

Therefore, using Assumption A, there exists $B_{n,1}(t) > 0$ such that

$$\left|F(x)\frac{\alpha\left(e^{(g-\beta)s}-1\right)}{g-\beta+\alpha x\left(e^{(g-\beta)s}-1\right)} - F(y)\frac{\alpha\left(e^{(g-\beta)s}-1\right)}{g-\beta+\alpha y\left(e^{(g-\beta)s}-1\right)}\right| \leq B_{n,1}(t)|x-y|.$$

To prove that (b) holds, it remains to prove that for any $n \in \mathbb{N}$, there exists $B_{n,2}(t) < \infty$ such that for all $0 \leq x \leq y \leq n$, and $0 \leq s \leq t$,

$$\int_0^1 \int_0^\infty (1-\theta)\left|x\mathbf{1}_{\{u\leq f_2(x,s,\theta)\}} - y\mathbf{1}_{\{u\leq f_2(y,s,\theta)\}}\right| du\kappa(d\theta) \leq B_{n,2}(t)|y-x|.$$

But for any $|x|, |y| \leq n$,

$$\left|x\mathbf{1}_{u\leq f_2(x,s,\theta)} - y\mathbf{1}_{u\leq f_2(y,s,\theta)}\right| \leq n\left|\mathbf{1}_{u\leq f_2(x,s,\theta)} - \mathbf{1}_{u\leq f_2(y,s,\theta)}\right| + |x-y|\mathbf{1}_{u\leq f_2(y,s,\theta)}.$$

Hence

$$\int_0^\infty \left|x\mathbf{1}_{u\leq f_2(x,s,\theta)} - y\mathbf{1}_{u\leq f_2(y,s,\theta)}\right| du \leq n|f_2(x,s,\theta) - f_2(y,s,\theta))| + |x-y|f_2(y,s,\theta).$$

Next,

$$\begin{aligned}|f_2(x,s,\theta)-f_2(y,s,\theta))| \\ \leq 2\left|(\alpha x+\beta)\frac{g-\beta+\alpha\theta x\left(e^{(g-\beta)s}-1\right)}{g-\beta+\alpha x\left(e^{(g-\beta)s}-1\right)}-(\alpha y+\beta)\frac{g-\beta+\alpha\theta y\left(e^{(g-\beta)s}-1\right)}{g-\beta+\alpha y\left(e^{(g-\beta)s}-1\right)}\right| \\ \leq 2\alpha|x-y|+2\alpha\beta\left(e^{|g-\beta|t}-1\right)|x-y|,\end{aligned}$$

and

$$\int_0^\infty \left|x\mathbf{1}_{u\leq f_2(x,s,\theta)}-y\mathbf{1}_{u\leq f_2(y,s,\theta)}\right| \leq B_{n,2}(t)|x-y|,$$

where $B_{n,2}(t)=2\alpha+2\alpha\beta\left(e^{|g-\beta|t}-1\right)+2(\alpha n+\beta)$ and condition (b) holds with $B_n(t)=g+B_{n,1}(t)+1/2B_{n,2}(t)$.

It remains to check that (c) is satisfied *i.e.* for all $0\leq s\leq t$, $z\geq 0$ and $u\geq 0$ that $x\mapsto x+h(x,s,z,u)$ is nondecreasing, where $h(x,s,z,u)=z\mathbf{1}_{\{u\leq f_3(x,s,z)\}}$ and that there exists $C_n(t)$ such that for all $0\leq x,y,\leq n$ and $0\leq s\leq t$,

$$|\sigma(x)-\sigma(y)|^2+\int_0^\infty\int_0^\infty \left(|l(x,y,s,z,u)|\wedge(l(x,y,s,z,u))^2\right)du\pi(dz)\leq C_n(t)|x-y|,$$

where $l(x,y,s,z,u)=h(x,s,z,u)-h(y,s,z,u)$. First, notice that for all $0\leq s\leq t$ and $z\geq 0$, $x\mapsto f_3(x,s,z)$ is nondecreasing thanks to Assumption G so that $x\mapsto x+h(x,s,z,u)$ is nondecreasing. Next,

$$\begin{aligned}\int_0^\infty \left(|l(x,y,s,z,u)|\wedge(l(x,y,s,z,u))^2\right)du &= \int_0^\infty \left(z\wedge z^2\right)\left|\mathbf{1}_{\{u\leq f_3(x,s,z)\}}-\mathbf{1}_{\{u\leq f_3(y,s,z)\}}\right|du \\ &= \left(z\wedge z^2\right)|f_3(x,s,z)-f_3(y,s,z)|\,.\end{aligned}$$

Moreover,

$$\begin{aligned}&|f_3(x,s,z)-f_3(y,s,z)| \\ &\leq |p(x)-p(y)|+\alpha z\left|e^{(g-\beta)s}-1\right|\left|\frac{p(x)}{g-\beta+\alpha x\left(e^{(g-\beta)s}-1\right)}-\frac{p(y)}{g-\beta+\alpha y\left(e^{(g-\beta)s}-1\right)}\right| \\ &\leq |p(x)-p(y)|+\alpha z\left(e^{|g-\beta|t}+1\right)\left(\left|\frac{p(x)-p(y)}{g-\beta}\right|+\alpha\left(e^{|g-\beta|t}+1\right)|xp(x)-yp(y)|\right).\end{aligned}$$

Finally, we get the desired inequality using that p is locally Lipschitz, $\int_0^\infty z^2\pi(dz)<\infty$ and that σ is 1/2-Hölderian. □

4.5. **Proofs of Section 3.2.**

Proof of Proposition 3.3. This proof is very similar to the proof of Theorem 2.3 and Corollary 2.4. The only modifications are due to the time-inhomogeneity of the auxiliary process, and to the fact that the time interval is restricted to $[0,t]$. We proceed by coupling to overcome these two difficulties.

First, we prove that Theorem 2.3 and Corollary 2.4 still hold if the rate of positive jumps depends on time and jump size. Let $t>0$ and consider a process X solution of the SDE (2.1) with the difference that the rate at which a jump of size $z>0$ occurs, when the time s is smaller than t is denoted by $p_t(X_{s^-},s,z)$ and satisfies for $x\geq 0$

$$p(x)\leq p_t(x,s,z)\leq p(x)(1+\mathfrak{c}_t z), \tag{4.28}$$

where $\mathfrak{c}_t$ is a finite and positive constant depending on t. As before, we define

$$G_a^{(s)}(x) := (a-1)g(x) - a(a-1)\frac{\sigma^2(x)}{x^2} - r(x)\int_0^1 \left(\theta^{1-a}-1\right)\kappa(d\theta)$$
$$- p_t(x,s,z)\int_{\mathbb{R}_+}\left((zx^{-1}+1)^{1-a} - 1 - (1-a)zx^{-1}\right)\pi(dz)$$

Using the upper bound in (4.28) we can show as in page 20 that under (3.2)

$$\limsup_{0^+}\left(x^{-2}\int_0^\infty p_t(x,s,z)z^2\left(\int_0^1 (1+zx^{-1}v)^{-1-a}(1-v)dv\right)\pi(dz)\right) < \infty.$$

Then, applying Itô's Formula with jumps we can check that Lemma 4.1 still holds with $G_a^{(s)}(X_s)$ instead of $G_a(X_s)$. The proofs of Theorem 2.3 and Corollary 2.4 are thus unchanged and the results hold also for processes whose rate of positive jumps satisfy (4.28).

Let us first consider point *i)* and introduce $\widetilde{Y}$ the unique strong solution to

$$\widetilde{Y}_t = \widetilde{Y}_0 + \int_0^t g\widetilde{Y}_s ds + \int_0^t \sqrt{2\sigma^2(\widetilde{Y}_s)}dB_s + \int_0^t \int_0^{f_3(\widetilde{Y}_{s^-},t-s,z)}\int_{\mathbb{R}_+} z\widetilde{Q}(ds,dx,dz)$$
$$+ \int_0^t \int_0^{2(\alpha \widetilde{Y}_{s^-}+\beta)}\int_0^1 (\theta-1)\widetilde{Y}_{s^-}N(ds,dz,d\theta),$$

where the Brownian motion and the Poisson random measures are the same as in (3.4), and by convention we decide for instance that $f_3(x,u,z)=0$ if $u\le 0$. Notice that for all $y\ge 0$, $0\le s\le t$, $0\le\theta\le 1$, and $z\ge 0$,

$$f_1(y,s)\ge g, \quad f_2(y,s,\theta)\le 2(\alpha y+\beta),$$

and $f_3(x,t-s,z)$ is increasing in x (thanks to Assumption G). In particular this implies that if $\widetilde{Y}$ is a solution with $\widetilde{Y}_0 = Y_0^{(t)}$, then $\widetilde{Y}_s \le Y_s^{(t)}$ for any s smaller than t. But $\widetilde{Y}$ satisfies assumptions of point *i)* of Corollary 2.4, and thus does not reach 0 in finite time. We deduce that $Y^{(t)}$ does not reach 0 before time t.

Now, let us prove point *ii)*. First notice that there exist two positive constants A_t and B_t depending on t and such that for any $x>0$ and $s\le t$, the function f_1 defined on page 11 satisfies

$$f_1(x,s) \le g + \left(2\sigma^2(x)/x + \mathbb{E}\left[\mathcal{Z}^2\right]p(x)/x\right)\frac{A_t}{B_t+xA_t} =: \bar{g}_t(x).$$

Now, introduce $(\bar{Y}_s, s\ge 0)$ as the unique strong solution to

$$\bar{Y}_s = Y_0^{(t)} + \int_0^s \bar{g}_t\left(\bar{Y}_u\right)\bar{Y}_u du + \int_0^s \sqrt{2\sigma^2\left(\bar{Y}_u\right)}dB_u + \int_0^s\int_0^{\bar{r}(\theta)}\int_0^1 (\theta-1)\bar{Y}_{u^-}N(du,dz,d\theta)$$
$$+ \int_0^s\int_0^\infty\int_0^{f_3(\bar{Y}_{s^-},t-s,z)} z\widetilde{Q}(du,dz,dx), \qquad (4.29)$$

with again $f_3(x,u,z)=0$ if $u\le 0$, and where B, N and $\widetilde{Q}$ are the same as in (3.4) and for all $x\ge 0$, $s\ge 0$ and $\theta\in[0,1]$,

$$\bar{r}(\theta) := 2\theta\beta \le f_2(x,s,\theta).$$

Then, for all $0\le s\le t$, $\bar{Y}_s \ge Y_s^{(t)}$. As a consequence, if we introduce

$$\bar{\tau}^-(0) := \inf\{s\ge 0 : \bar{Y}_s = 0\},$$

and prove that for any $0 < v \leq t$,

$$\mathbb{P}(\bar{\tau}^-(0) < v) > 0, \tag{4.30}$$

it will end the proof. Indeed (4.30) implies that

$$\mathbb{P}(\tau_t^-(0) < v) \geq \mathbb{P}(\bar{\tau}^-(0) < v) > 0.$$

To prove (4.30), we apply point *iii)* of Corollary 2.4 to the process $\bar{Y}$. Notice that here, unlike in Corollary 2.4, the division rate $\bar{r}$ depends on θ. However, according to Lemma 2.2 the dependence in θ in the division rate can be removed by considering a new Poisson point measure N' with a modified fragmentation kernel so that all the results derived above still hold. It ends the proof. □

Proof of Proposition 3.4. We prove the convergence of the auxiliary process by verifying a Foster-Lyapunov inequality and a minoration condition, both stated in Lemma 4.3 below. Those standard conditions were exhibited in [29] as an extension of [15] to time-inhomogeneous processes. The Foster-Lyapunov inequality (Condition *i)* in Lemma 4.3) ensures that

$$\mathbb{E}_x\left[V\left(Y_s^{(t)}\right)\right] \leq e^{-as}V(x) + \frac{d}{a}\left(1 - e^{-as}\right),$$

where a and d are positive constants, so that the process is brought back to the sublevel sets of V. The minoration condition (Condition *ii)* in Lemma 4.3) ensures some type of irreducibility of the process on those sublevel sets. Combining those two conditions yields the convergence of the auxiliary process (see Proposition 3.2 in [29]).

Let

$$V(x) = x \quad \text{for } x \in \mathbb{R}_+.$$

Lemma 4.3. *Under the assumptions of Proposition 3.4, we have the following:*

i) There exist $a, d > 0$ such that for all $0 \leq s \leq t$ and $x \in \mathbb{R}_+^*$,

$$\mathcal{A}_s^{(t)}V(x) \leq -aV(x) + d.$$

ii) There exists $R > 2da^{-1}$ such that for all $r < s \leq t$, there exist $\alpha_{s-r} > 0$ and a probability measure ν on $\mathbb{R}_+$ such that for all Borel set A of $\mathbb{R}_+$,

$$\inf_{x \leq R} \mathbb{P}\left(Y_s^{(t)} \in A \middle| Y_r^{(t)} = x\right) \geq \alpha_{s-r}\nu(A).$$

Proof. i) We have

$$\begin{aligned}\mathcal{A}_s^{(t)}V(x) &= f_1(x, t-s)x - \int_0^1 f_2(x, t-s, \theta)x(1-\theta)\kappa(d\theta) \\ &\leq gx + \left(2\frac{\sigma(x)^2}{x^3} + \mathbb{E}\left[\mathcal{Z}^2\right]\frac{p(x)}{x^3} - 2\alpha\mathbb{E}\left[\Theta(1-\Theta)\right]\right)x^2.\end{aligned}$$

According to (3.5), there exists $A > 0$ and $a > 0$ such that for all $x > A$,

$$\left(2\frac{\sigma(x)^2}{x^3} + \mathbb{E}\left[\mathcal{Z}^2\right]\frac{p(x)}{x^3} - 2\alpha\mathbb{E}\left[\Theta(1-\Theta)\right]\right)x < -(a+g),$$

where we have used the inequality

$$\mathbb{E}\left[\Theta(1-\Theta^5)\right] = \mathbb{E}\left[\Theta(1-\Theta)(1+\Theta+\Theta^2+\Theta^3+\Theta^4)\right] \leq 5\mathbb{E}\left[\Theta(1-\Theta)\right].$$

Hence, there exists $d > 0$ such that for every $x \geq 0$,

$$\mathcal{A}_s^{(t)}V(x) \leq -ax + d.$$

ii) Let $R > 2da^{-1}$, where a, d are given in *i)*. We will prove the minoration condition with $\nu_s = \delta_0$, where δ_0 is the Dirac mass at 0. Consider again $\bar{Y}$, defined as the unique strong solution to the SDE (4.29). We recall that $\bar{Y}_s \geq Y_s^{(t)}$, for all $s \leq t$. Therefore for all $r < s \leq t$ and all Borel set A of $\mathbb{R}_+$,

$$\mathbb{P}\left(Y_s^{(t)} \in A \middle| Y_r^{(t)} = x\right) \geq \mathbb{P}\left(Y_s^{(t)} = 0 \middle| Y_r^{(t)} = x\right) \delta_0(A) \geq \mathbb{P}\left(\bar{Y}_s = 0 \middle| \bar{Y}_r = x\right) \delta_0(A).$$

Next, notice that if $\bar{Y}^1, \bar{Y}^2$ are two solutions to (4.29) with respective initial conditions at time r satisfying $\bar{Y}_r^1 \leq \bar{Y}_r^2$, then $\bar{Y}_s^1 \leq \bar{Y}_s^2$ for all $r \leq s \leq t$. Hence, for all $x \leq R$,

$$\mathbb{P}\left(\bar{Y}_s = 0 \middle| \bar{Y}_r = x\right) = \mathbb{P}\left(\bar{Y}_{s-r} = 0 \middle| \bar{Y}_0 = x\right) \geq \mathbb{P}\left(\bar{Y}_{s-r} = 0 \middle| \bar{Y}_0 = R\right).$$

Finally, using point *iii)* of Corollary 2.4 on $\bar{Y}$, there exists $\alpha_{s-r} > 0$ such that

$$\mathbb{P}\left(\bar{Y}_{s-r} = 0 \middle| \bar{Y}_0 = R\right) > \alpha_{s-r},$$

which ends the proof. □

4.5.1. *Proof of Theorem 3.5.* We now prove the convergence of the process corresponding to the trait of a sampling to the limit of the auxiliary process as t goes to infinity. It is a direct application of [29, Corollary 3.4]. Assumptions A, B, C in [29] are satisfied thanks to Assumptions B, H and (3.3). We proved that Assumption D in [29] is verified in Lemma 4.3. It remains to check that Assumptions E and F in [29] are satisfied. Note that in our case, the function $c(x)$ defined in [29, Equation 3.3] is equal to $\max(g, \beta)$ and the first point of Assumption E in [29] is satisfied because $g + \beta \neq 0$.

Next, we set some notations, introduced in [29]. For all $x, y \geq 0$ and $s \geq 0$, we define

$$\varphi_s(x, y) = \sup_{t \geq s} \frac{m(x, 0, s) m(y, s, t)}{m(x, 0, t)},$$

and for all measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ and $x \geq 0$,

$$Jf(x) = 2 \int_0^1 f(\theta x) f((1-\theta)x) \kappa(d\theta).$$

The next lemma amounts to check the second point of Assumption E in [29].

Lemma 4.4. *For all $x \geq 0$,*

$$\sup_{t \geq 0} \mathbb{E}_x \left[r\left(Y_t^{(t)}\right) J\left((1 \vee V(\cdot)) \varphi_t(x, \cdot)\right) \left(Y_t^{(t)}\right) \right] < \infty.$$

Proof. Note that if $x = 0$, $Y_t^{(t)} = 0$ almost surely for all $t \geq 0$. Therefore, we only need to consider $x > 0$. First, notice that for all $t \geq 0$ $x > 0$, and $y \geq 0$,

$$\varphi_t(x, y) \leq \frac{\left(1 + \frac{\alpha x}{|g-\beta|}\right)\left(1 + \frac{\alpha y}{|g-\beta|}\right)}{\min\left(\frac{\alpha x}{|g-\beta|}, 1\right)}.$$

Next, for all $x > 0$,

$$\begin{aligned}
&\mathbb{E}_x \left[r\left(Y_t^{(t)}\right) J\left((1 \vee V(\cdot)) \varphi_t(x, \cdot)\right) \left(Y_t^{(t)}\right) \right] \\
&\leq \left(\frac{|g-\beta| + \alpha x}{\min(\alpha x, |g-\beta|)} \right)^2 \mathbb{E}_x \left[\left(\alpha Y_t^{(t)} + \beta\right) 2 \int_0^1 \left(1 \vee \theta Y_t^{(t)}\right) \left(1 \vee (1-\theta) Y_t^{(t)}\right) \left(1 + \frac{\alpha Y_t^{(t)}}{|g-\beta|}\right)^2 \kappa(d\theta) \right].
\end{aligned}$$

For all $k \geq 0$, we define

$$f_k^{(t)}(x, s) = \mathbb{E}_x \left[\left(Y_s^{(t)}\right)^k \right]$$

and we end the proof of the lemma by showing that,

$$\sup_{t\geq 0}\sup_{s\leq t} f_5^{(t)}(x,s)<\infty.$$

According to Itô's formula, we have for $k\geq 2$,

$$\begin{aligned}
f_k^{(t)}(x,s) =&k\int_0^s \mathbb{E}_x\left[\left(Y_u^{(t)}\right)^k f_1\left(Y_u^{(t)},t-u\right)\right]du+k(k-1)\int_0^s \mathbb{E}_x\left[\sigma^2\left(Y_u^{(t)}\right)\left(Y_u^{(t)}\right)^{k-2}\right]du\\
&+\int_0^s\int_{\mathbb{R}_+}\mathbb{E}_x\left[f_2\left(Y_u^{(t)},t-u,\theta\right)\left(Y_u^{(t)}\right)^k(\theta^k-1)\right]\kappa(d\theta)du\\
&+\int_0^s\int_{\mathbb{R}_+}\mathbb{E}_x\left[f_3\left(Y_u^{(t)},t-u,z\right)\left(\left(Y_u^{(t)}+z\right)^k-\left(Y_u^{(t)}\right)^k-kz\left(Y_u^{(t)}\right)^{k-1}\right)\right]\pi(dz)du.
\end{aligned}$$

Differentiating with respect to s and using that for all $x\geq 0$ and $s\geq 0$, $x^kf_1(x,s)\leq gx^k+2\sigma(x)^2x^{k-2}+\mathbb{E}[\mathcal{Z}^2]p(x)x^{k-2}$, $f_2(x,s,\theta)\geq 2\theta\alpha x$ for all $\theta\in[0,1]$ and $f_3(x,s,z)\leq (x+z)p(x)/x$ for all $z\geq 0$, and applying Taylor's formula with integral remainder, we obtain

$$\begin{aligned}
\partial_s f_k^{(t)}(x,s) \leq& gk\mathbb{E}_x\left[\left(Y_s^{(t)}\right)^k\right]+k\mathbb{E}_x\left[\left((k+1)\sigma^2\left(Y_s^{(t)}\right)+\mathbb{E}[\mathcal{Z}^2]p\left(Y_s^{(t)}\right)\right)\left(Y_s^{(t)}\right)^{k-2}\right]\\
&-\int_{\mathbb{R}_+}\mathbb{E}_x\left[2\alpha\left(Y_s^{(t)}\right)^{k+1}\theta(1-\theta^k)\right]\kappa(d\theta)\\
&+k(k-1)\int_{\mathbb{R}_+}\int_0^z(z-u)\mathbb{E}_x\left[\frac{p\left(Y_s^{(t)}\right)}{Y_s^{(t)}}\left(Y_s^{(t)}+u\right)^{k-2}\left(Y_s^{(t)}+z\right)\right]du\pi(dz).
\end{aligned}$$

Moreover, for all $y\geq 0$,

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_0^z(z-u)\frac{p\left(y\right)}{y}\left(y+u\right)^{k-2}\left(y+z\right)du\pi(dz)\\
&\qquad\leq \int_{\mathbb{R}_+}z^2\frac{p(y)}{y}\left(y+z\right)^{k-1}\pi(dz)=\frac{p(y)}{y^3}\int_{\mathbb{R}_+}y^2z^2\sum_{l=0}^{k-1}\binom{k-1}{l}y^lz^{k-1-l}\pi(dz)\\
&\qquad\qquad=\frac{p(y)}{y}\int_{\mathbb{R}_+}z^2\sum_{l=0}^{k-2}\binom{k-1}{l}y^lz^{k-1-l}\pi(dz)+\frac{p(y)}{y^3}\mathbb{E}\left[\mathcal{Z}^2\right]y^{k+1}.
\end{aligned}$$

Combining the last two inequalities, we get

$$\partial_s f_k^{(t)}(x,s)\leq k(A_t^{(k)}+B_t^{(k)}+C_t^{(k)}),$$

with

$$A_t^{(k)}=g\mathbb{E}_x\left[(Y_s^{(t)})^k\right],\quad B_t^{(k)}=\mathbb{E}_x\left[H(k,Y_s^{(t)})(Y_s^{(t)})^{k+1}\right],$$

$$C_t^{(k)}=(k-1)\int_{\mathbb{R}_+}\mathbb{E}_x\left[\frac{p(Y_s^{(t)})}{Y_s^{(t)}}\sum_{l=0}^{k-2}\binom{k-1}{l}(Y_s^{(t)})^lz^{k+1-l}\right]\pi(dz),$$

where

$$H(k,y)=(k+1)\frac{\sigma^2(y)}{y^3}+k\mathbb{E}[\mathcal{Z}^2]\frac{p(y)}{y^3}-\frac{2\alpha}{k}\mathbb{E}\left[\Theta(1-\Theta^k)\right].$$

To end the proof we consider the case $k=5$. According to (3.5) and using that σ and p are continuous (Assumption A), there exist $C_1,C_2>0$ and $A>0$ such that for all $y\geq 0$,

$$\begin{aligned}
H(5,y)y^6=H(5,y)y^6\mathbf{1}_{\{y>A\}}+H(5,y)y^6\mathbf{1}_{\{y\leq A\}}&\leq -C_1y^6\mathbf{1}_{\{y>A\}}+C_2\mathbf{1}_{\{y\leq A\}}\\
&\leq -C_1y^6+C_1A^6+C_2.
\end{aligned}$$

Moreover, $\limsup_{0+} p(x)/x < \infty$ according to Assumption E, $\limsup_{\infty} p(x)/x^3 < \infty$ thanks to (3.5) and $\mathbb{E}\left[\mathcal{Z}^6\right] < \infty$, which yields

$$C_t \leq C_3(f_5^{(t)}(x,s)+1),$$

for some $C_3 \geq 0$. Combining the last two inequalities, there exist $D_1, D_2 > 0$ such that

$$\partial_s f_5^{(t)}(x,s) \leq D_1(f_5^{(t)}(x,s)+1) - D_2 f_6^{(t)}(x,s).$$

Applying Jensen inequality, we have $f_6^{(t)}(x,s) \geq f_5^{(t)}(x,s)^{6/5}$. Finally, we obtain

$$\partial_s f_5^{(t)}(x,s) \leq F\left(f_5^{(t)}(x,s)\right),$$

with $F(y) = D_1(y+1) - D_2 y^{1+1/5}$. Any solution to the equation $y' = F(y)$ is bounded by $y(0) \vee x_0$, where $x_0 = (5D_1/6D_2)^5$ and so is $f_5^{(t)}(x,\cdot)$. It ends the proof. □

Finally, we need to control the value of the second moment of the population size relatively to the square of its mean. It corresponds to Assumption F in [29].

Lemma 4.5. *Suppose that Assumptions B, C, E, F hold. Then for all $x > 0$,*

$$\mathbb{E}_{\delta_x}\left[N_t^2\right] \sim m^2(x,0,t)\left(\mathbf{1}_{\{\beta<g\}} + \mathbf{1}_{\{g\leq\beta\}}\frac{(g-\beta-\alpha x)^2 + (g-\beta)^2(1-2\alpha x/(g-2\beta))}{(g-\beta-\alpha x)^2}\right), \quad (t\to\infty).$$

Proof. According to Itô's formula, we have for all $t \geq 0$ and $x > 0$,

$$\begin{aligned}\mathbb{E}_{\delta_x}\left[N_t^2\right] &= 1 + \int_0^t \mathbb{E}_{\delta_x}\left[(\alpha x e^{gs} + \beta N_s)(1+2N_s)\right] ds \\ &= 1 + \frac{\alpha x}{g}\left(e^{gt}-1\right) + 2\alpha x\int_0^t e^{gs}\mathbb{E}_{\delta_x}[N_s]ds + \beta\int_0^t \mathbb{E}_{\delta_x}[N_s]ds + 2\beta\int_0^t \mathbb{E}_{\delta_x}\left[N_s^2\right] ds.\end{aligned}$$

Using (3.3), we get

$$\begin{aligned}\mathbb{E}_{\delta_x}\left[N_t^2\right] =& e^{2\beta t} + \frac{2\alpha x}{g-\beta}\left(1-\frac{\alpha x}{g-\beta}\right)\left(e^{(g+\beta)t}-e^{2\beta t}\right) + \frac{\alpha^2x^2}{(g-\beta)^2}\left(e^{2gt}-e^{2\beta t}\right) \\ &+ \frac{\alpha x}{g-2\beta}\frac{g}{g-\beta}\left(e^{gt}-e^{2\beta t}\right) + \left(1-\frac{\alpha x}{g-\beta}\right)\left(e^{2\beta t}-e^{\beta t}\right).\end{aligned} \tag{4.31}$$

Moreover,

$$m(x,0,t)^2 = e^{2\beta t} + \frac{2x\alpha}{g-\beta}\left(e^{(g+\beta)t}-e^{2\beta t}\right) + \frac{x^2\alpha^2}{(g-\beta)^2}\left(e^{2gt}-2e^{(g+\beta)t}+e^{2\beta t}\right).$$

We end the proof by letting t go to infinity. □

We checked that all required assumptions to apply [29, Corollary 3.4] are satisfied. This ends the proof of Theorem 3.5.

4.6. **Proofs of Section 3.3.**

Proof of Proposition 3.6. We first prove point *ii)* and assume that $\limsup_{0+}\sigma^2(y)/y < \infty$. The first step consists in proving that (2.13) holds for Y, but a direct application of Theorem 2.5 is not possible because of the time-inhomogeneity of the process Y. Therefore, we couple Y with a process $(\hat{Y}_s, s \geq 0)$ defined as the unique strong solution to

$$\begin{aligned}\hat{Y}_s =& Y_0^{(t)} + \int_0^s \hat{g}(\hat{Y}_u)\hat{Y}_u du + \int_0^s \sqrt{2\sigma^2(\hat{Y}_u)}dB_u + \int_0^s\int_0^{\hat{r}(\hat{Y}_u,\theta)}\int_0^1(\theta-1)\hat{Y}_u N(du,dz,d\theta) \\ &+ \int_0^s\int_0^\infty\int_0^{f_3(\hat{Y}_u,t-s,z)} z\widetilde{Q}(du,dz,dx),\end{aligned} \tag{4.32}$$

where $f_3(y,t-s,z)=0$ for $s\geq t$, B, N and $\tilde{Q}$ are the same as in (3.4) and for $x,s\geq 0$, $0\leq\theta\leq 1$,

$$\hat{g}(x):=g+\frac{\alpha}{\beta-g+\alpha x}\left(2\frac{\sigma^2(x)}{x}+\mathbb{E}\left[\mathcal{Z}^2\right]\frac{p(x)}{x}\right)\geq f_1(x,s),\quad \hat{r}(x,\theta):=2\theta(\alpha x+\beta)\leq f_2(x,s,\theta),$$

where the first inequality holds because $\beta>g$. Then, for all $t\geq 0$ and $0\leq s\leq t$, $Y_s^{(t)}\leq\hat{Y}_s$. In particular, for all $t\geq 0$,

$$Y_t^{(t)}\leq\hat{Y}_t. \tag{4.33}$$

According to Lemma 2.2, there exists a Poisson point measure N' on $\mathbb{R}_+\times[0,1]\times\mathbb{R}_+$ with intensity $du\otimes\hat{\kappa}(d\theta)\otimes dx$ such that $\hat{Y}$ is also a strong pathwise solution to

$$\hat{Y}_s=Y_0^{(t)}+\int_0^s\hat{g}\left(\hat{Y}_u\right)\hat{Y}_u du+\int_0^s\sqrt{2\sigma^2\left(\hat{Y}_u\right)}dB_u+\int_0^s\int_0^{r(\hat{Y}_u)}\int_0^1(\theta-1)\hat{Y}_uN'(du,dz,d\theta)$$
$$+\int_0^s\int_0^\infty\int_0^{f_3(\hat{Y}_u,t-s,z)}z\widetilde{Q}(du,dz,dx). \tag{4.34}$$

As the jump rate f_3 in the integral with respect to $\widetilde{Q}$ depends on z and time, we need to prove that Lemma 4.2 still holds under these modifications. Following the same steps as in the proof of this lemma for $\hat{Y}$ instead of X we have, when $T_1>t_0$,

$$\ln(\hat{Y}_{t\wedge T_1})=\ln(\hat{Y}_{t_0})+\int_{t_0}^{t\wedge T_1}\hat{g}(\hat{Y}_s)ds-\int_{t_0}^{t\wedge T_1}\frac{\sigma^2(\hat{Y}_s)}{\hat{Y}_s^2}ds+M_{t\wedge T_1}+\int_{t_0}^{t\wedge T_1}r(\hat{Y}_s)\left(\int_0^1\ln\theta\hat{\kappa}(d\theta)\right)ds$$
$$+\int_{t_0}^{t\wedge T_1}f_3(\hat{Y}_s,t-s,z)\left[\ln(\hat{Y}_s+z)-\ln(\hat{Y}_s)-z/\hat{Y}_s\right](1+z/\hat{Y}_s)\pi(dz)ds, \tag{4.35}$$

where $(M_{s\wedge T_1},s\geq 0)$ is a martingale. Let us check that (B0) is satisfied. Let $\hat{\Theta}$ be a random variable with law $\hat{\kappa}$. We have

$$\hat{g}(x)+r(x)\mathbb{E}\left[\ln\hat{\Theta}\right]=g+\frac{\alpha}{\beta-g+\alpha x}\left(2\sigma^2(x)/x+\mathbb{E}\left[\mathcal{Z}^2\right]p(x)/x\right)+2(\alpha x+\beta)\mathbb{E}\left[\Theta\ln\Theta\right],$$

and (B0) is satisfied using (3.5) and the fact that the function

$$x\in(0,1]\mapsto 5\ln 1/x-(1-x^5)$$

is nonnegative (which is obtained by the study of the function's derivative). Moreover, notice that the dependence on z and on time for the jump rate in the integral with respect to $\widetilde{Q}$ does not modify the proof of Lemma 4.2, as the last term in (4.35) is still negative. If we look at the proof of (2.13) for $\hat{Y}$, we see that we only need to check that there exists $\mathfrak{d}>0$ such that

$$0<\inf_{0\leq x\leq x_0+\mathfrak{d}}(\hat{g}(x)+r(x))\leq\sup_{0\leq x\leq x_0+\mathfrak{d}}(\hat{g}(x)+r(x))<\infty$$

which is true under our assumptions. Hence (2.13) holds for $\hat{Y}$. Applying Theorem 3.5 to the function $F(X_{t+s}^u,s\leq T)=\mathbf{1}_{\{X_{t+T}^u=0\}}$ concludes the proof of point *ii)*.

Let us now prove point *i)*. If $g<\beta$ and $\limsup_{0+}\sigma^2(y)/y<\infty$, $\hat{Y}$ is still well-defined as a strong solution of (4.34) and satisfies (B0) according to (3.5). Then, from Theorem 2.5, $\hat{Y}_t$ converges in law to $\hat{Y}_\infty$ with distribution given by (2.11) and using (4.33) we obtain for all $K>0$

$$\mathbb{P}(\hat{Y}_\infty>K)\geq\lim_{t\to\infty}\mathbb{P}\left(Y_t^{(t)}>K\right).$$

By Markov inequality,

$$\mathbb{P}_{\delta_x}\left(\frac{\sum_{u\in V_t}\mathbf{1}_{\{X_t^u>K\}}}{N_t}>\varepsilon\right)\leq \varepsilon^{-1}\frac{\mathbb{E}_{\delta_x}\left[\sum_{u\in V_t}\mathbf{1}_{\{X_t^u>K\}}\right]}{m(x,0,t)}=\varepsilon^{-1}\mathbb{P}\left(Y_t^{(t)}>K\right), \quad (4.36)$$

where the last equality comes from (3.1) applied to the function

$$F((X_s^u, s\leq t)) = \mathbf{1}_{\{X_t^u>K\}}.$$

Finally, taking the limit in (4.36) in t and K yields the result. In the case where the assumptions of Proposition 3.4 are satisfied, according to this latter proposition, $Y_t^{(t)}$ converges to a nondegenerate random variable, which implies

$$\lim_{K\to\infty}\lim_{t\to\infty}\mathbb{P}(Y_t^{(t)}>K)=0.$$

We conclude as before using (4.36). □

4.7. **Proofs of Section 3.4.**

Proof of Proposition 3.7. Let us begin with the proof of point *ii)*. From Proposition 2.6*i)*, we have

$$Y_t\to 0 \quad a.s.$$

and combining Equation (3.1) and the fact that $\mathbb{E}_{\delta_x}[N_t]=e^{\beta t}$, we obtain that

$$\mathbb{E}_{\delta_x}\left[\frac{\sum_{u\in V_t}\mathbf{1}_{\{X_t^u>\varepsilon\}}}{e^{\beta t}}\right]\to 0, \quad (t\to\infty).$$

Moreover, the fact that $(N_t, t\geq 0)$ is a Yule process also entails that $N_t e^{-\beta t}$ converges in probability to $W(1)$, exponential random variable with parameter 1, when t goes to infinity (see [1] Chap.III Section 5). Hence, we have

$$\frac{\sum_{u\in V_t}\mathbf{1}_{\{X_t^u>\varepsilon\}}}{N_t}=\frac{\sum_{u\in V_t}\mathbf{1}_{\{X_t^u>\varepsilon\}}}{e^{\beta t}}\times\frac{1}{N_t e^{-\beta t}}\xrightarrow{\mathbb{P}} 0,$$

when t tends to infinity. It ends the proof of point *ii)*.

We now prove point *iii)*. Applying Equation (2.13) in Theorem 2.5 to the process Y, we obtain that

$$\mathbb{P}(Y_t\neq 0)\to 0, \quad (t\to\infty).$$

From this, similarly as for the proof of point *ii)* we obtain

$$\frac{\sum_{u\in V_t}\mathbf{1}_{\{X_t^u=0\}}}{N_t}\to 1 \quad \text{in probability}, \quad (t\to\infty).$$

To prove that the convergence of *iii)* holds almost surely we follow the proof of Theorem 4.2.(i) in [6] where the authors prove that the convergence in probability implies the almost sure convergence by using a technical result on Yule processes (see [6, Lemma 4.3]).

We end with the proof of point *i)*. Applying Corollary 2.7.*iii)* to Y, we obtain that

$$\liminf_{t\to\infty} Y_t e^{-\Lambda_{\rho(t)}}=W,$$

with $\mathbb{P}(W>0)>0$ and where Λ is a Lévy process with drift η and $\rho(t)\geq t$. Writing, for $\varepsilon>0$,

$$Y_t e^{-(\eta-\varepsilon)t}=Y_t e^{-\Lambda_{\rho(t)}}e^{\Lambda_{\rho(t)}-(\eta-\varepsilon)t},$$

and noticing that $\Lambda_{\rho(t)}-(\eta-\varepsilon)t$ goes to ∞ when t goes to ∞, we get

$$\mathbb{P}_x\left(\liminf_{t\to\infty} Y_t e^{-(\eta-\varepsilon)t}>0\right)>0,$$

and thus by Fatou Lemma

$$\liminf_{t\to\infty} \mathbb{P}_x\left(Y_t e^{-(\eta-\varepsilon)t} > 0\right) > 0.$$

Hence, using Equation (3.1) we obtain

$$\liminf_{t\to\infty} \mathbb{E}_{\delta_x}\left[\frac{\sum_{u\in V_t} \mathbf{1}_{\{X_t^u e^{-(\eta-\varepsilon)t}>0\}}}{e^{\beta t}}\right] > 0.$$

Now notice that Cauchy-Schwarz inequality yields

$$\begin{aligned}\mathbb{E}^2_{\delta_x}\left[\frac{\sum_{u\in V_t} \mathbf{1}_{\{X_t^u e^{-(\eta-\varepsilon)t}>0\}}}{e^{\beta t}}\right] &\leq \mathbb{E}_{\delta_x}\left[\left(\frac{\sum_{u\in V_t} \mathbf{1}_{\{X_t^u e^{-(\eta-\varepsilon)t}>0\}}}{N_t}\right)^2\right]\mathbb{E}_{\delta_x}\left[\left(\frac{N_t}{e^{\beta t}}\right)^2\right]\\ &\leq \mathbb{E}_{\delta_x}\left[\frac{\sum_{u\in V_t} \mathbf{1}_{\{X_t^u e^{-(\eta-\varepsilon)t}>0\}}}{N_t}\right]\mathbb{E}_{\delta_x}\left[\left(\frac{N_t}{e^{\beta t}}\right)^2\right],\end{aligned}$$

where the last inequality comes from the fact that the term in the first expectation in the right-hand side is smaller than one. Noticing that the last expectation converges to 1 as t goes to infinity we obtain

$$0 < \liminf_{t\to\infty} \mathbb{E}^2_{\delta_x}\left[\frac{\sum_{u\in V_t} \mathbf{1}_{\{X_t^u e^{-(\eta-\varepsilon)t}>0\}}}{e^{\beta t}}\right] \leq \liminf_{t\to\infty} \mathbb{E}_{\delta_x}\left[\frac{\sum_{u\in V_t} \mathbf{1}_{\{X_t^u e^{-(\eta-\varepsilon)t}>0\}}}{N_t}\right].$$

This ends the proof of the first part of point *i)*. The proof of the second point of point *i)* follows the proof of Theorem 4.2.(ii) in [6]. □

Appendix A. Auxiliary process, case $g = \beta$

In the case $g = \beta$, the mean number of individuals takes the form

$$m(x,s,t) = (1+\alpha x(t-s))e^{g(t-s)}.$$

As a consequence,

$$\begin{aligned}\widehat{\mathcal{G}}_s^{(t)} f(x) = &\left(gx + \left(2\sigma^2(x) + x\mathbb{E}\left[\mathcal{Z}^2\right]\right)\frac{\alpha(t-s)}{1+x\alpha(t-s)}\right)f'(x) + \sigma^2(x)f''(x)\\ &+ x\int_{\mathbb{R}_+}(f(x+z)-f(x)-zf'(x))\left(1+\frac{\alpha z(t-s)}{1+\alpha x(t-s)}\right)\pi(dz),\end{aligned}$$

$$\widehat{r}_s^{(t)}(x) = (\alpha x+\beta)\left(1+\frac{1}{1+x\alpha(t-s)}\right) \quad \text{and} \quad \widehat{\kappa}_s^{(t)}(x,d\theta) = \mathbf{1}_{\{0\leq\theta\leq 1\}}\frac{2+2\alpha\theta x(t-s)}{2+\alpha x(t-s)}\kappa(d\theta).$$

Moreover, the functions $(f_i, 1\leq i\leq 3)$ take the form

$$f_1(y,s) := g + \left(2\frac{\sigma^2(y)}{y} + \mathbb{E}\left[\mathcal{Z}^2\right]\right)\frac{\alpha s}{1+\alpha y s}, \quad f_2(y,s,\theta) := 2(\alpha y+\beta)\frac{1+\alpha\theta y s}{1+\alpha y s},$$

and

$$f_3(y,s,z) := p(y)\left(1+\frac{\alpha z s}{1+\alpha y s}\right).$$

Proof of Proposition 3.1. We keep the notation of the proof in the case $g \neq \beta$ and only provide the computations which differ from this case.

$$\left|\sigma^2(x)\frac{\alpha s}{1+\alpha x s} - \sigma^2(y)\frac{\alpha s}{1+\alpha y s}\right| \leq \widetilde{B}_{n,1}(t)|x-y|,$$

with $\widetilde{B}_{n,1}(t) = \alpha t\left(1 + \alpha t\left(\sup_{1\leq x\leq n}\sigma^2(x) + n\right)\right)$,

$$\mathbb{E}\left[\mathcal{Z}^2\right]\alpha s\left|\frac{x}{1+\alpha xs} - \frac{y}{1+\alpha ys}\right| \leq \widetilde{B}_{n,2}(t)|x-y|,$$

with $\widetilde{B}_{n,2}(t) = \mathbb{E}\left[\mathcal{Z}^2\right]\alpha t$ and

$$\int_0^\infty \left|x\mathbf{1}_{u\leq f_2(x,s,\theta)} - y\mathbf{1}_{u\leq f_2(y,s,\theta)}\right| du \leq \widetilde{B}_{n,3}(t)|x-y|,$$

with $\widetilde{B}_{n,3}(t) = 2\alpha + 4(\alpha n + \beta)\alpha t + 2(\alpha n + \beta)$ and condition (b) holds with $\widetilde{B}_n(t) = g + 2\widetilde{B}_{n,1}(t) + \widetilde{B}_{n,2}(t) + 1/2\widetilde{B}_{n,3}(t)$.

Moreover,

$$|f_3(x,s,z) - f_3(y,s,z)| \leq |p(x) - p(y)| + \alpha zt|xp(x) - yp(y)|,$$

which ends the proof. □

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